# (c-)AND: A new graph model

Mauricio Soto¹\* and Christopher Thraves Caro²\*\*

DIM, Universidad de Chile, Chile
GSyC, Universidad Rey Juan Carlos, Spain

Abstract. In this document, we study the scope of the following graph model: each vertex is assigned to a box in a metric space and to a representative element that belongs to that box. Two vertices are connected by an edge if and only if its respective boxes contain the opposite representative element. We focus our study on the case where boxes (and therefore representative elements) associated to vertices are spread in the Euclidean line. We give both, a combinatorial and an intersection characterization of the model. Based on these characterizations, we determine graph families that contain the model (e. g., boxicity 2 graphs) and others that the new model contains (e. g., rooted directed path). We also study the particular case where each representative element is the center of its respective box. In this particular case, we provide constructive representations for interval, block and outerplanar graphs. Finally, we show that the general and the particular model are not equivalent by constructing a graph family that separates the two cases.

## 1 Introduction

A disk graph is a graph where the set of vertices corresponds to a collection of points that belong to a metric space and an edge connects two vertices if and only if their corresponding points are at a distance of at most a parameter r. An important application of disk graphs is in the area of sensor networks. Sensor networks are networks formed by sensor nodes, little devices deployed in a geographic area with monitor purposes. Sensors communicate with each other via a radio channel. Every sensor covers with its radio signal a communication area around it and two sensors communicate with each other when they are placed within each other communication area. In an ideal model, the communication area of a sensor is a circle. Therefore, in the same ideal model, if every sensor covers equally sized communication areas, the network formed by sensors is a disk graph. That explains why researchers have used disk graphs to represent sensor networks, particularly unit disk graphs [FCFM09] or some variations [KWZ08].

<sup>\*</sup> Email: mausoto@dim.uchile.cl. Gratefully acknowledges the support of CONICYT, Apoyo al retorno de investigadores desde el extranjero grant 82130059.

<sup>\*\*</sup> Email: cbthraves@gsyc.es. Christopher Thraves Caro is supported by Spanish MICINN grant Juan de la Cierva, Comunidad de Madrid grant S2009TIC-1692 and Spanish MICINN grant TIN2008-06735-C02-01.

Nevertheless, it is difficult to find such an ideal situation in a real deployment, mainly due to physical or geographical restrictions. For instance, when the deployment area is irregular, the communication area of a sensor might be shrunken in one direction due to an obstacle, while, in the opposite direction, the area is free of any obstacle. On the other hand, some sensors may have directional antennas which produce communication areas that are far from being a circle, or that place the sensor location far from the *center* of its communication area. Therefore, the existence of a communication link between two sensors is not determined by the distance between them, neither by the intersection of their communication areas. In fact, one has to be sure that the communication areas *cover* the opposite sensor.

Consequently, we propose a new graph family that aims to include the different topologies that may be created due to those restrictions. Consider a set S and an element  $p \in S$  as a representative element of S. Consider now a graph where each vertex corresponds to a pair (S, p) and an edge between two vertices exists if and only if the set associated with a vertex contains the representative element of its fellow and vice versa. According to this definition, nonempty intersection between two sets is not enough to guarantee the existence of their corresponding edge. Moreover, when the sets belong to a metric space, there is no positive distance between two representative elements that guarantees the existence of their corresponding edge. Therefore, this definition differs from disk graphs, as well as from intersection graphs.

In this document, we consider the family induced by the above definition when sets are boxes in an Euclidean metric space. We aim to understand the properties of such a graph family. We focus our study on the case where boxes and representative elements associated to vertices are spread in the Euclidean line. We study the extent of this definition as a graph family. We provide an intersection model and a combinatorial characterization of the model. Additionally, we tackle the subfamily of graphs where all representative elements are the center of its respective boxes.

## 2 Definitions

We consider graphs that are finite, connected, undirected, loopless and without parallel edges. For a graph G = (V, E), we denote by V(G) and E(G) the set of vertices and edges, respectively. When the graph under consideration is clear, we use only V and E. The edge  $\{u,v\}$  is denoted by uv. If  $uv \in E(G)$  we say that v is a neighbor of u and vice versa. The set of neighbors of u is denoted by  $\mathcal{N}(u)$ . Additionally, the closed neighborhood of u is defined as  $\mathcal{N}[u] := \mathcal{N}(u) \cup \{u\}$ .

A box in the d-dimensional Euclidean space is the Cartesian product of d closed intervals. A box B is described as the set  $B = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : L_i \leq x_i \leq R_i\}$ , where  $L_i$  and  $R_i$  denote the extreme points of the interval in the i-th dimension. The center of a box is the Cartesian product of the centers of the intervals in each dimension of the box. Namely, the center of the box B as defined above is the point  $((L_1 + R_1)/2, (L_2 + R_2)/2, \dots, (L_d + R_d)/2)$ .

**Definition 1** (AND-realization). An AND-realization of a graph G in the d-dimensional Euclidean space is a collection of pairs  $\{(B_v, p_v) : v \in V(G)\}$  where each vertex v is associated to a d-dimensional box  $B_v$  and to a representative element  $p_v \in B_v$ , such that:

$$uv \in E(G) \Leftrightarrow (p_v \in B_u) \land (p_u \in B_v).$$

A central AND-realization or c-AND-realization of a graph is an AND-realization in which each representative element  $p_v$  is the center of its box  $B_v$ .

We denote by AND(d) the set of graphs that admit an AND-realization in the d-dimensional Euclidean space. The subset of AND(d) that contains the graphs that admit a c-AND-realization in the d-dimensional Euclidean space is denoted by c-AND(d). For simplicity, all along this document, we use notation (c-)AND when we say something that concerns to both classes c-AND and AND.

We mainly study sets AND(1) and c-AND(1). In this context, a box  $B_u$  becomes an interval in the Euclidean line that we denote by its extreme points [L(u), R(u)]. Any (c-)AND(1)-realization can be modified so that it maintains the graph it represents. For a given realization  $\mathcal{R} = \{([L(u), R(u)], p_u)\}_{u \in V(G)},$  we define  $\delta$ -translation and  $\sigma$ -scaling of  $\mathcal{R}$  as the realizations  $\{([L(u) + \delta, R(u) + \delta], p_u + \delta)\}_{u \in V(G)}$  and  $\{([\sigma \cdot L(u), \sigma \cdot R(u)], \sigma \cdot p_u)\}_{u \in V(G)}$ , respectively.

Any (c-)AND(1)-realization of a graph induces a natural ordering of its vertices following its representative elements, i.e, v < u according to a (c-)AND(1)-realization if and only if  $p_v < p_u$  in that (c-)AND(1)-realization. In order to properly define this order, each representative element must differ from each other. Nevertheless, it is easy to see that any (c-)AND(1)-realization can be modified to fulfill this property.

Given an ordering  $\pi$  of the vertices of a graph G, we denote by  $<_{\pi}$  the total order induced by  $\pi$ . That is,  $u <_{\pi} v$  if u appears before v in  $\pi$ . The extreme vertices of an order  $<_{\pi}$  are the vertices placed at the first and last position according to  $<_{\pi}$ . Given a vertex u, we denote by  $\ell_{\pi}(u)$  and  $\rho_{\pi}(u)$  the leftmost and rightmost neighbors of u in the order, i.e.,  $\ell_{\pi}(u) = \{v \in \mathcal{N}[u] : v <_{\pi} w \forall w \in \mathcal{N}[u], w \neq v\}$  and  $\rho_{\pi}(u) = \{v \in \mathcal{N}[u] : w <_{\pi} v \forall w \in \mathcal{N}[u], w \neq v\}$ .

## 3 Related work and our contributions

We compare the introduced AND family of graphs with other graph classes. Therefore, we refer the reader to an excellent survey authored by Brandstädt et al. [BLS99] that contains a description of almost all graph families involved in this document. This survey also presents containment relations between classes, graphs that separate one class form another, and priceless information in this area. A second excellent book that we refer to the reader is authored by J. Spinrad [Spi03]. This book deals with efficient graph representation. For several graph classes, this book considers such questions as existence of good representations, algorithms for finding representations, questions of characterization in terms of representation, and how the representation affects the complexity of optimization problems.

There exists a vast amount of interesting literature related with the graph families that we mention in this document [MM99,Gol04,Rob69a,GT04]: geometric, outerplanar, interval, max-tolerance and boxicity 2 graphs. The study of intersection graphs dates back a long way. For instance, the fact that all graph can be represented as an intersection graph was proved by Marczewski and Sur in [SMS45] and by Erdös et al. in [EGP64]. Related with particular graph families, the notion of boxicity of a graph is introduced in [Rob69b]. On the other hand, the notion of book embedding of a graph is introduced in [BK79] where the authors present some first properties and relations with other invariants such as thickness, genus, and chromatic number. An intersection model for max-tolerance is introduced in [KKLS06]. Such a model was of great utility at the moment of determining the NP-Hardness of recognition problem for max-tolerance graphs. Finally, the book [GT04] surveys results related with (max-)tolerance graphs.

The AND(1) family has been addressed very recently in parallel to our work via totally independent way. T. Hixon in his Master thesis [Hix13] studies the family of cyclic segment graphs. This family corresponds to the intersection graphs of segments that lie on lines tangent to a parabola and no two segments are parallel. In his thesis, Hixon also works on the subclass called *hook graphs*, in which all segment in the representation need to be tangent to the parabola. The author proves that a graph is a hook graph if and only if it is the intersection graph of a set of axis aligned rectangles in the plane such that the top left corner of each rectangle lies on a unique point on the diagonal. Such result is equivalent to the particular case d=1 of Theorem 1 in this document. The author also proves a combinatorial characterization for hook graphs which is is equivalent to Theorem 2 in our work. This result has been also obtained independently by Feuilloley [Feu] in the study of the gap between Minimum Hitting Size problem and Maximum Independent Set problem for the AND(1) family. Based on this characterization Hixon proves that interval, outerplanar, and 2-directional orthogonal ray graphs (2DORG) are all hook graphs. We extend these results by proving that interval and outerplanar graphs are not only AND(1) but also c-AND(1) graphs. On the other hand, we extend the fact that interval graphs belong to AND(1) by proving that a larger family, rooted directed path graphs, belongs to AND(1). Moreover, T. Hixon proves that in a hook graph two nonadjacent vertices cannot be connected by three induced disjoint paths of length larger than 4. We prove indeed that such paths, in the case of c-AND(1), cannot be all longer than 3. The author gives also polynomial algorithms for the Weighted Maximum Clique problem and Weighted Maximum Independent Set problem; and approximations for the Chromatic Number and Clique Covering Number. Finally, according to Hixon, the AND(1) family is addressed by Cantanzaro et al. [CCH<sup>+</sup>] in the context of DNA sequences. Nevertheless, as far as we know, their work has not been published. Hence, it has been impossible for us compare our results with theirs.

#### 3.1 Our contributions

The main contribution of this document is the study of the two graph families: AND(d) and c-AND(d). We study the one-dimensional version (c-)AND(1) of the families in which the position of the representative elements induce an order of the vertices.

- We give a characterization of both AND(d) and c-AND(d) families via an intersection model in Subsection 4.1.
- We give a characterization of the And(1) family via a combinatorial characterization of the possible orders of its vertices in any And(1)-realization in Subsection 4.2.
- We give a construction of AND(1)-realizations for INTERVAL BIGRAPHS and ROOTED DIRECTED PATH graphs, and c-AND(1)-realizations for INTERVAL, BLOCK and OUTERPLANAR graphs in Subsection 4.2 and Section 5, respectively.
- Finally, in Section 6, we show differences between families AND(1) and c-AND(1), proving that in the first case two non-adjacent vertices cannot be connected via three disjoint paths with edge-length strictly larger than 3. While, in the second case, two non-adjacent vertices cannot be connected via three disjoint paths with edge-length strictly larger than 2.

## 4 Characterizations for AND graphs

In this section, we show that AND(d) graphs can be represented by an intersection model. Besides, we give a combinatorial characterization for the set of graphs that admit an AND(1)-realization. From these characterizations, we obtain containment relations with other non trivial graph classes.

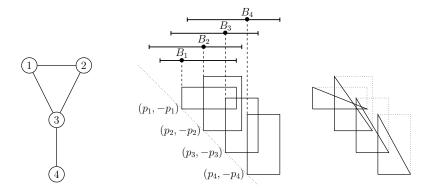
#### 4.1 Intersection graph characterization for AND graphs

We show in this subsection that graphs in the AND(d) class can be represented as the intersection graph of boxes in the  $2 \cdot d$ -dimensional Euclidean space.

**Theorem 1.** A graph G belongs to AND(d) if and only if G is the intersection graph of boxes in the  $2 \cdot d$  Euclidean space, and each box can be described as  $\times_{i=1}^{d}([p_i, R_i] \times [-p_i, -L_i])$  with  $p_i, R_i, L_i > 0$  for all  $i \in \{1, ..., d\}$ .

Proof. Let G be a graph that belongs to AND(d). Consider a realization  $\{(B_v, p_v)\}_{v \in V}$  of G, where  $B_v = \times_{i=1}^d [L_i(v), R_i(v)]$  and  $p_v = \times_{i=1}^d p_{(v,i)}$ . W.l.o.g, assume  $L_i(v) > 0$  for all  $v \in V$  and  $i \in \{1, ..., d\}$ . For each  $v \in V$ , we define the box  $B'_{(v,i)}$  in the Euclidean plane as  $[p_{(v,i)}, R_i(v)] \times [-p_{(v,i)}, -L_i(v)]$ . Finally, let  $S_v$  be the  $2 \cdot d$ -dimensional box defined as the cartesian product of boxes  $B'_{(v,i)}$ , that is  $S_v = \times_{i=1}^d ([p_{(v,i)}, R_i(v)] \times [-p_{(v,i)}, -L_i(v)])$ .

that is  $S_v = \times_{i=1}^d ([p_{(v,i)}, R_i(v)] \times [-p_{(v,i)}, -L_i(v)])$ . Consider two vertices  $u, v \in V$ , then  $uv \in E$  if and only if  $p_{(u,i)} \in [L_i(v), R_i(v)]$  and  $p_{(v,i)} \in [L_i(u), R_i(u)]$  for all  $i \in \{1, ..., d\}$ . For a fixed i, let us assume



**Fig. 1.** An example of a graph in AND(1) (left) with its intersection model with boxes (center) and triangles (right hand side).

 $p_{(u,i)} < p_{(v,i)}$ . Thus,  $p_{(v,i)} - p_{(u,i)} \le \min\{R_i(u) - p_{(u,i)}, p_{(v,i)} - L_i(v)\}$  or equivalently  $B'_{(u,i)} \cap B'_{(v,i)} \ne \emptyset$ . Hence, vertices u and v are adjacent if and only if  $S_u \cap S_v \ne \emptyset$ .

For the one-dimensional case (d=1), Theorem 1 states that AND(1) graphs correspond exactly to the intersection graphs of boxes in the Euclidean plane with its left-lower corner lying on the diagonal  $\mathcal{L}: x+y=0$  (an example is shown in Figure 1).

Kaufmann et al. in [KKLS06] proved that max-tolerance graphs correspond to the class of intersection of isosceles, axis parallel, right triangles (or lower halves of a square). A different representation of AND(1) graphs can be obtained by keeping the left lower half of the boxes in the intersection model (an example is shown in Figure 1). Particularly, when this intersection model is applied to c-AND(1) graphs, we obtain an intersection model of isosceles, axis parallel, right triangles (or lower halves of a square). Therefore, the following corollary holds.

Corollary 1. c-AND(1)  $\subset$  MAX-TOLERANCE.

## 4.2 A combinatorial characterization for the AND(1) graphs

We recall that any AND(1)-realization of a graph induces a natural ordering of its vertices by considering their respective representative elements. This ordering needs to have different representative elements in order to be totally defined. Nevertheless, it is easy to see that any (c-)AND(1)-realization can be modified to fulfill this property.

**Definition 2.** (R-order) Given a graph G that belongs to AND(1) and an AND(1)-realization  $\mathcal{R}$  of G such that all representative elements are different. The  $\mathcal{R}$ -order of the set V, denoted by  $<_{\mathcal{R}}$ , is the total order induced by the representative elements. That is, for any pair of vertices u and  $v: u <_{\mathcal{R}} v \Leftrightarrow p_u < p_v$ .



Fig. 2. Graphic representation of the four point condition for AND(1).

Consider an  $\mathcal{R}$ -order of a graph G and two vertices  $u <_{\mathcal{R}} v$  in V. If vertex u has a neighbor y after v ( $v <_{\mathcal{R}} y$ ) and v has a neighbor x before u ( $x <_{\mathcal{R}} u$ ). Then, vertices u and v are mutually contained in its corresponding intervals. Thus, vertices u and v must be connected. Indeed, this property characterizes graphs that belong to the set AND(1). Therefore, we introduce the following definition for any ordering of the set of vertices of a graph.

**Definition 3.** Given a graph G = (V, E) and an order  $<_{\pi}$  of its set of vertices. We say that  $<_{\pi}$  satisfies the four point condition for AND(1) if and only if for every quadruplet of vertices x, u, v, y, it holds:

If 
$$x <_{\pi} u <_{\pi} v <_{\pi} y$$
 and  $xv, uy \in E \Rightarrow uv \in E$ .

Figure 2 shows a graphic representation of the four point condition for AND(1).

We prove that for any graph G the existence of an ordering of its set of vertices that satisfies the four point condition for AND(1) is necessary and sufficient to decide if G belongs to AND(1).

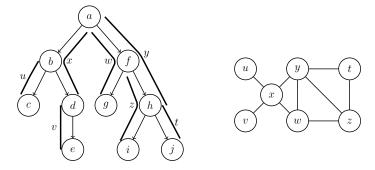
**Theorem 2.** A graph G belongs to AND(1) if and only if there exists an ordering of its set of vertices that satisfies the four point condition for AND(1).

*Proof.* As we have seen previously, the four point condition is necessary for any AND(1)-realization of G. For the converse, let  $<_{\pi}$  be any ordering of the vertices of G which satisfies the four point condition.

Let  $\mathcal{R}_{\pi}$  be a realization constructed in the following way: representative elements  $p_v$  are embedded in the Euclidean line arbitrarily but respecting the order  $<_{\pi}$ . For each  $v \in V$ , we define  $B_v$  as the interval covering from the leftmost to the rightmost neighbors of v according to  $<_{\pi}$ , that is  $B_v = [\ell_{\pi}(v), \rho_{\pi}(v)]$ .

In order to verify that  $\mathcal{R}_{\pi}$  is an AND(1)-realization of G, consider an edge  $uv \in E$  with  $u <_{\pi} v$ . By definition of  $\mathcal{R}_{\pi}$ , it holds that  $u \in B_v$  and  $v \in B_u$ . On the other hand, if  $u \in B_v$  and  $v \in B_u$ , then there exist vertices  $y \in \mathcal{N}(u)$  and  $x \in \mathcal{N}(v)$  such that  $x <_{\pi} u <_{\pi} v <_{\pi} y$ . Thus, vertices u and v are neighbors by the four point condition.

Remark 1. Note that the above construction allows us to place the representative elements of the vertices in the integers ranging from 1 to n. Hence, any AND(1) graph can be represented as the collection of  $B_v = \{[\ell_{\pi}(v), \rho_{\pi}(v)], p_v\}$  for all  $v \in V$ , where  $\ell_{\pi}(v), \rho_{\pi}(v)$  and  $p_v$  are integers ranging from 1 to n. On the other hand, in any AND(1)-realization, adjacency between two vertices u and v can



**Fig. 3.** At the right hand side, this figure shows an example of a rooted directed path graph. At the left hand side, this figure shows a rooted directed path representation of the graph. An inverse DFS on that tree is (jihgfedcba) inducing the following order of the vertices of the graph: (tzywvxu).

be tested by performing four operations in order to check  $p_u \in [\ell_{\pi}(v), \rho_{\pi}(v)]$  and  $p_v \in [\ell_{\pi}(u), \rho_{\pi}(u)]$ . Therefore, we can conclude that the family of graphs AND(1) admits an *implicit representation* as defined in [Spi03], which is: an *implicit representation* of a graph G is defined as a representation of G that assigns  $O(\log n)$  bits to each vertex, such that there is an adjacency testing algorithm that decides adjacency between two vertices u and v based only on the bits stored at vertices u and v.

The four point condition is a useful tool to recognize graph families that belong to the set AND(1) as well as families that do not belong to it. We now present three graph families that belong to AND(1), for which we show the existence of an ordering that satisfies the four point condition.

A graph is a rooted directed path graph (also known as directed path graphs) if it has an intersection model consisting of directed paths in a rooted directed tree, where every arc is oriented from the root to the leaves. Figure 3 shows an example of a rooted path tree and its corresponding intersection model.

#### Corollary 2. ROOTED DIRECTED PATH graphs belong to the set AND(1).

Proof. In order to prove the Corollary, we give an ordering of the vertices of any rooted directed path graph G such that the four point condition for And(1) is satisfied. Let G = (V, E) be a rooted directed path graph and T = (K, F) be a tree with an intersection model of G consisting of directed paths in T. For each v in V, let us denote by  $K_v$  the directed path in T corresponding to vertex  $v \in V$ . Note that for every v,  $K_v$  is a subset of K. We order K, the vertex set of T, using an inverse DFS ordering, i.e., first order K according to a DFS (cf. [Gol04]), and then inverse the ordering. Let us denote by  $\pi: K \to \{1, 2, \ldots, |K|\}$  the permutation given by the ordering of K, i.e.,  $\pi(k) = i$  when k is the i-th vertex in the inverse DFS ordering. We define as well the following notation:  $\pi(K_v) = \{\pi(k) : k \in K_v\}$ . Since K has an inverse DFS ordering and the fact that

G is a rooted directed path graph, for every vertex  $v \in V$ ,  $\pi(K_v)$  is increasing when  $K_v$  is traversed bottom-up in the tree.

Now, vertex set V is ordered according to the minimum value of  $\pi(K_v)$ . If required, break ties randomly. In order to conclude the proof of the Corollary, we show now that the ordering of V satisfies the four point condition for AND(1). The proof is by contradiction. Assume that there exist four vertices in V such that they violate the condition. I.e., consider four vertices x < u < v < y in V such that  $\{xv, uy\} \subseteq E$  but  $uv \notin E$ . Since x < u < v < y in the ordering of V, it holds  $\min \pi(K_x) \le \min \pi(K_u) \le \min \pi(K_v)$  in the ordering of K.

Given that  $xv \in E$ , it holds that  $\pi(K_x) \cap \pi(K_v) \neq \emptyset$ . Since x < u < v,  $\min \pi(K_u) \leq \min \{\pi(K_x) \cap \pi(K_v)\}$ . Furthermore, due to  $uv \notin E$ , for every  $j \in \pi(K_u)$  it holds that  $j < \min \pi(K_v)$ . Now, since  $uy \in E$ , hence  $\min \pi(K_y) < \min \pi(K_v)$ . Therefore, we obtain y < v, which is a contradiction.

## Corollary 3. Outerplanar graphs belong to And (1).

Proof. In order to prove that outerplanar graphs belong to AND(1), let us recall the definition of page embedding of a graph (cf. [BK79]). A k-page embedding, or book embedding, of a graph G consists in an linear ordering of the vertices of G which are drawn on a line (the spine of the book) together with a partition of the edges into k pages such that two edges in the same page do not cross. The pagenumber of a graph is the smallest k for which the graph has a k-page embedding. In [Bil92], Bilski proved that outerplanar graphs are exactly the graphs with pagenumber one. Therefore, for any outerplanar graph there exists an ordering of its vertices in which the edges do not cross. Such an ordering satisfies the four point condition for AND(1).

In contrast to the previous corollary, the four point condition helps as well to discard a graph from the And(1) set.

**Corollary 4.** Let G be a graph such that all pairs of vertices  $u, v \in V$  have at least two non adjacent common neighbors. Then G does not belong to AND(1).

*Proof.* The proof is by contradiction. Let us assume that there exists a graph G that belongs to AND(1) such that all pairs of vertices  $u, v \in V$  have at least two non adjacent common neighbors. In order to reach the contradiction, we give four vertices in V that do not satisfy the four point condition for AND(1). Let  $\mathcal{R}$  be an AND(1)-realization for G. Consider the two extreme vertices of  $\mathcal{R}$ , say vertices x and y. There exist two vertices u and v such that  $uv \notin E$  and  $\{xu, xv, uy, vy\} \subseteq E$ . Now, for any order of vertices u and v, it holds that the quadruplet x, u, v, y does not satisfy the four point condition for AND(1).

# 5 Subclasses of c-AND(1)

In this section, we establish the relation between the c-AND(1) family and other well-known graph families. Particularly, we enhance the result by Hixon [Hix13]

by showing that interval and outerplanar graphs belong not only to AND(1) but also to c-AND(1).

**Theorem 3.** The set of Interval graphs is a subset of c-And(1).

Proof. Let G be an interval graph. In [Ola91] Olariu proves that for any interval graph there exists an ordering  $<_{\pi}$  of its vertex set V such that for all triplet  $u, v, w \in V$  with  $u <_{\pi} v <_{\pi} w$  and  $uw \in E$  then  $uv \in E$ . Moreover, this order can be obtained in linear time. Consider such an ordering for the vertex set V. For the sake of simplicity, we relabel vertices in V from 1 to n according to the ordering  $<_{\pi}$ .

We construct a c-AND(1)-realization of G greedily. At the i-th step, we include the vertex i in the c-AND(1)-realization. The inclusion is performed in such a way that, at the end of the step i, it holds, for all j, k, w in  $\{1, \ldots, i\}$ , that:

```
1. p_{k-1} < p_k 3. L(j) < p_{\ell(j)} 2. \rho(j) <_{\pi} \rho(k) \Rightarrow R(j) < R(k). 4. \rho(k) <_{\pi} j \Leftrightarrow R(k) < p_j.
```

Condition 1 ensures that representative elements are placed according to order  $<_{\pi}$ . Condition 2 ensures that right extremes of intervals are in the same order than the values of  $\rho(\cdot)$ . Finally, conditions 3 and 4 guarantee that the partial realization at the end of step i corresponds to the subgraph induced by vertices  $1, 2, \ldots, i$ . Thus, at the end of the construction a c-AND(1)-realization of G is obtained.

At the first step, vertex 1 is included so that  $p_1 = 0$  and [L(1), R(1)] = [-1, 1]. At the end of the first step all conditions are satisfied. Let us suppose that all conditions hold at the end of the step i-1. We include vertex i in the c-AND(1)-realization in two phases:

- First, we set the position of representative element  $p_i$  respecting conditions 1 and 4. That is, the representative element is placed after  $p_{i-1}$  and it is contained only by intervals associated to its previous neighbors.
- Second, we set the interval associated to i such that it contains all its previous neighbors, according to condition 3. Finally, we modify, if necessary, the interval of previous vertices in order to satisfy conditions 2.

For the first phase we remark that if two vertices j, k have labels smaller than i and  $j \notin \mathcal{N}(i) \land k \in \mathcal{N}(i)$  then  $\rho(j) < i \leq \rho(k)$ . Therefore, by condition 2, we have that R(j) < R(k). Thus, by defining  $L = \max\{R(j) : j \notin \mathcal{N}(i)\}$  and  $R = \min\{R(k) : k \in \mathcal{N}(i)\}$ , it holds L < R. Notice that in between L and R there might exist some representatives elements. Hence, by setting  $p_i$  as  $(\max\{p_{i-1}, L\} + R)/2$ , conditions 1 and 4 hold and first phase is concluded.

In order to set the extremes of interval  $B_i$ , let define  $P_i = \{j < i : \rho(j) < \rho(i)\}$ , the set of all vertices having its last neighbor before i. We recall that condition 2 imposes that R(j) < R(i) for all vertex j in  $P_i$ . If R' denotes the  $\max\{R(j): j \in P_i\}$  then it must holds that R' < R(i). On the other hand, the interval  $B_i$  must contain  $p_{\ell(i)}$  so that condition 3 is satisfied. Then, let define  $r_i$  as  $\max\{p_i - p_{\ell(i)}, R' - p_i\} + 1$ . We set  $L(i) = p_i - r_i$  and  $R(i) = p_i + r_i$  so all conditions are satisfied for vertices in  $P_i$ . However, condition 2, does not

necessary hold for vertices that do not belong to  $P_i \cup \{i\}$ . To overcome this problem, we extend the intervals of those vertices by  $2r_i$ . That is, we re-define  $B_j$  as  $[L(j)-r_i,R(j)+r_i]$  for all  $j \notin P_i \cup \{i\}$ . Thus, since  $R(i)=p_i+r_i < R(j)+r_i$  condition 2 is satisfied for all vertices in V.

The rest of the section aims to prove that Outerplanar graphs belong to c-And(1). We first show that cycles belong to c-And(1). Moreover, we show that any realization of a cycle has a specific structure. Secondly, we construct a procedure to combine biconnected components and show how to "glue" two different cycles by an edge.

**Lemma 1.** Let  $C_n$  be a cycle of length n, then  $C_n$  belongs to c-AND(1). Furthermore, let  $\mathcal{R}$  be an AND(1)-realization of  $C_n$  and  $\pi$  be the permutation induced by  $<_{\mathcal{R}}$ . Then, there exists a clockwise (or anticlockwise) labeling  $l: V \to \{1, 2, \ldots, n\}$  such that:

- 1. Extreme vertices are adjacent and  $\pi(l^{-1}(1)) = 1 \wedge \pi(l^{-1}(n)) = n$ .
- 2. For all  $u \in V$ ,  $|l(u) \pi(u)| \le 1$
- 3. If  $\mathcal{R}$  is a c-AND(1)-realization then for all  $u \in V$ ,  $l(u) = \pi(u)$ .

Proof. Let  $C_n$  be a cycle. We prove that  $C_n$  belongs to c-AND(1) by constructing a realization. Let us label the vertex set V clockwise starting in an arbitrary vertex. Given  $0 < \epsilon < 1$ , we associate to each vertex  $i \in \{2, \ldots, n-1\}$  the interval  $([i-(1+\epsilon),i+(1+\epsilon)]$  and the representative element  $p_i=i$ . Extreme vertices are assigned to pars (interval, representative element)  $([2-n-\epsilon,n+\epsilon],1)$  and  $([1-\epsilon,2n-1+\epsilon],n)$ , respectively. It is easy to check that the previous defined realization is actually a c-AND(1)-realization for  $C_n$ .

Consider an AND(1)-realization  $\mathcal{R}$  of the cycle  $C_n$ . If n=3 the representative elements are always in a (anti-)clockwise order. Assume then that n>3. We define a clockwise (or anticlockwise) labeling l of V as follows: (1) the vertex with label 1 has the minimum value of  $p_u$ , i.e.,  $(\pi \circ l^{-1}(1) = 1)$  and, (2) the vertex with label 2 is the neighbor of 1 with the smaller position in the order:  $\pi \circ l^{-1}(2) < \pi \circ l^{-1}(n)$ .

Condition 1 is proved by contradiction. Note that by definition  $l^{-1}(1)$  is an extreme vertex. Hence, assume that  $l^{-1}(n)$  is not a extreme vertex. Define w as follows:  $p_{l^{-1}(n)} < p_w$  and  $l(w) \leq l(w')$  for all w' such that  $p_{l^{-1}(n)} < p_{w'}$ . By the definition of the labeling, it holds that l(w) > 2, moreover w has a neighbor placed between the vertices with labels 1 and n, which we denote by v. We conclude that quadruplet  $l^{-1}(1) <_{\mathcal{R}} v <_{\mathcal{R}} l^{-1}(n) <_{\mathcal{R}} w$  violates four point condition, which is a contradiction. Hence, vertex  $l^{-1}(n)$  is an extreme vertex.

We prove 1 and 1 greedily. First, let us introduce some definitions. We say that a vertex u satisfies the pre-condition if for all v such that  $p_v < p_u$  it holds  $l(v) < \pi(u)$ . Clearly, extreme vertices satisfy the pre-condition. Let w be a vertex that satisfies the pre-condition but such that  $l(w) \neq \pi(w)$ , then it must hold that  $l(w) > \pi(w)$ . Let us denote by v the vertex such that  $p_v < P_w$  and  $l(v') \leq l(v)$  for all v' such that  $p_{v'} < P_w$ . By the definition of w, it holds that l(v) < n - 1. We denote by v the neighbor of v with label l(v) + 1, thus v < x. Let v' be the vertex in between v and v with the maximum label. Since l(v) < l(x) < v then

w' must have a neighbor y (with label l(w')+1) such that  $p_x < p_y$ . Thus, by the four point condition in the quadruplet  $v <_{\mathcal{R}} w' <_{\mathcal{R}} x <_{\mathcal{R}} y$ , vertices w' and x must be neighbors. We conclude that w=w' and l(w)=l(x)+1=l(v)+2. Additionally, the vertex immediately after x, that is, in the position  $\pi(x)+1$ , satisfies the pre-condition.

As we state before, extreme vertices satisfy the pre-condition. Let w be the first vertex according to  $<_{\mathcal{R}}$  such that  $l(w) \neq \pi(w)$ . By definition, w satisfies the pre-condition. Thus, by the previous discussion, we have that  $l(w) = \pi(w) + 1$ . Furthermore, the next vertex in the ordering, say x, has label l(w) - 1 and then  $l(w) - \pi(w) = 1 \land l(x) - \pi(x) = -1$ . Furthermore, the next vertex in the ordering must satisfy the pre-condition. By iterating over vertices according to the order  $<_{\mathcal{R}}$ , we verify that Condition 1 holds. Finally, consider the case when  $\mathcal{R}$  is a c-AND(1)-realization. Let  $v <_{\mathcal{R}} w <_{\mathcal{R}} x <_{\mathcal{R}} y$  be the quadruplet previously constructed, where l(y) = l(w) + 1. If  $p_x$  is placed in the left half of the interval  $[p_v, p_y]$  then  $vw \in E$ , otherwise  $xy \in E$  which yields a contradiction. Thus, for all vertices in the c-AND(1)-realization  $l(w) = \pi(w)$ .

**Definition 4 (Safe vertex).** Let G be a graph in (c-)And(1). We say that a vertex  $v \in V$  is safe in G if there exists a (c-)And(1)-realization  $\mathcal{R} = \{(B_u, p_u)\}_{u \in V(G)}$  such that  $v \in B_w$  if and only if  $v = w \lor vw \in E(G)$ .

A safe vertex allows the union of two different biconnected components. This important property comes from the fact that in a realization where a vertex v is safe, the interval  $B_v$  can be extended as much as required without modifying the original graph.

**Lemma 2.** Consider two graphs  $G_1, G_2 \in (c\text{-})AND(1)$  and two vertices  $w_1 \in V(G_1)$  and  $w_2 \in V(G_2)$ . Let G be the graph obtained by identifying  $w_1$  and  $w_2$ . If  $w_2$  is safe in  $G_2$ , then it holds that  $G \in (c\text{-})AND(1)$ .

Proof. Let G be a graph obtained by the identification of vertices  $w_1$  and  $w_2$  of two different graphs  $G_1$  and  $G_2$ . Consider two (c-)AND(1)-realizations  $\mathcal{R}_1 = \{(B_u, p_u)\}_{u \in V(G_1)}$  and  $\mathcal{R}_2 = \{(B_u, p_u)\}_{u \in V(G_2)}$  of  $G_1$  and  $G_2$ , respectively, such that  $w_2$  is safe in  $\mathcal{R}_2$ . We denote by  $\Delta$  the minimum distance between  $p_{w_1}$  and the representative elements of its neighbors, that is  $\Delta = \min_{u \in \mathcal{N}(w_1)} \{|p_{w_1} - p_u|\}$ . Let B be an interval such that  $\bigcup_{v \in V(G_2)} \{B_v\} \subseteq B$  and denote by L its length. We construct the realization  $\mathcal{R}'_2 = \{(B'_v, p'_v)\}_{v \in V(G_2)}$  from  $\mathcal{R}_2$  by the following procedure:

- apply a  $(-p_{w_2})$ -translation in order to place the representative element of  $w_2$  in the origin,
- scale the realization by a factor  $\Delta/(2L)$ ,
- perform a  $(p_{w_1})$ -translation in order to equals the position of representatives elements of  $w_1$  and  $w_2$ .

Let  $B_w$  be the interval with center in  $p_{w_1}$  and of length equal to the maximum between  $B_{w_1}$  and  $B'_{w_2}$ . Then, let us define  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}'_2 \setminus \{(B_{w_1}, p_{w_1}), (B'_{w_2}, p'_{w_2})\} \cup (B_w, p_w)$ . We see that  $\mathcal{R}$  is a (c-)AND (1) realization for G. In fact, all edges

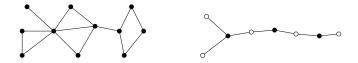


Fig. 4. This figure shows a graph (left hand side) and its block tree (right hand side). In the block tree representation, white vertices represent maximal biconnected components, while black vertices represent cut-vertices.

 $uv \in E(G_1) \cup E(G_2)$  are induced by  $\mathcal{R}$ . Furthermore by the definition of  $\mathcal{R}'_2$  and the fact that  $w_2$  is safe, no new edges are generated by  $\mathcal{R}$ .

Given a graph G, the block tree of G is the graph having two types of vertices: blocks and cut-vertices (cf. [BM07]). A block vertex represents a maximal biconnected component of G while cut-vertices are the articulation points between blocks. The edges of the block tree join blocks with cut-vertices. A block is adjacent to a cut-vertex if the block contains the cut-vertex. Figure 4 shows an example of a graph and its block tree.

**Theorem 4.** Let G be a connected graph and T be its block tree. If all maximal biconnected components of G belong to (c-)AND(1) and T can be rooted such that every cut-vertex is safe in its descendants, then G belongs to (c-)AND(1).

*Proof.* The proof follows directly from Lemma 2 by adding biconnected components of G in a breadth-first traversal BFS (cf. [Gol04]) order of T.

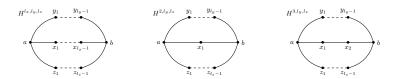
The previous result allows us to constructively obtain a realization of a graph by gluing the realization of its biconnected components. As a consequence, we obtain the following corollary.

**Corollary 5.** Block graphs, graphs in which all biconnected components induce a clique, belong to c-And(1).

An analogous result to Lemma 2 can be obtained to identify edges in two different cycles:

**Lemma 3.** Given two cycles  $C_n, C'_m$  and two edges  $uv \in E(C_n)$  and  $u'v' \in E(C'_m)$ , let G be the graph obtained by identifying uv and u'v'. Then,  $G \in c$ -AND(1).

Proof. For  $0 < \epsilon < 1$  we construct two c-AND(1)-realizations  $\mathcal{R}$  and  $\mathcal{R}'$  of  $C_n$  and  $C'_m$  respectively according to the procedure described in the proof of Theorem 1. Furthermore, we suppose that u and v are the extreme vertices of realization  $\mathcal{R}$  but u' and v' are not the extreme vertices of  $\mathcal{R}'$ . We perform a 1/(m-1)-scaling and a translation of  $\mathcal{R}$  so that the positions of representative elements of u and v equal those of u' and v' in  $\mathcal{R}'$ . The realization  $\mathcal{R} \cup \mathcal{R}' \setminus \{(B_{u'}, u'), (B_{v'}, v')\}$  is a c-AND(1)-realization for the graph G. Notice that this realization can be done with any vertex as an extreme (safe) vertex.



**Fig. 5.** This figure shows, from left to right, a graphic representation of (a general)  $H^{l_x,l_y,l_z}$ , and the particular cases of  $H^{2,l_y,l_z}$  and  $H^{3,l_y,l_z}$ .

**Theorem 5.** The set of OUTERPLANAR graphs is a subset of c-AND(1).

*Proof.* Maximal biconnected components of an outerplanar graph are dissections of a convex polygon, which belong to c-AND(1) by Lemma 3. The proof follows by gluing biconnected components according to Theorem 4.

## 6 Differences between AND(1) and c-AND(1)

In this section, we show the difference between AND(1) and c-AND(1) via graphs that belong to AND(1) but which does not belong to c-AND(1). We start with a remark upon the fact that the property of being part of AND(1) or c-AND(1) is hereditary, i.e., if a graph G belongs to (c-)AND(1) then every induced subgraph of G also belongs to (c-)AND(1). Indeed, if a graph G has a (c-)AND(1)-realization then the same realization is also a (c-)AND(1)-realization for every induced subgraph of G when the corresponding vertices are deleted. From the hereditary property, we define a graph G that does not belong to (c-)AND(1) as minimal with respect to (c-)AND(1) if and only if every proper induced subgraph of G does belong to (c-)AND(1). All graphs introduced here that separate AND(1) and c-AND(1) are minimal with respect to c-AND(1) and they are based in the following two definitions.

**Definition 5.** Let  $H^{l_x,l_y,l_z}$  be a finite graph that consists of two not neighboring vertices, say vertices a and b, together with three vertex disjoint paths that connect vertex a with vertex b. The three paths that connect vertex a with vertex b follow: path  $X = \{a = x_0, x_1, x_2, \ldots, x_{l_x-1}, x_{l_x} = b\}$ , path  $Y = \{a = y_0, y_1, y_2, \ldots, y_{l_y-1}, y_{l_y} = b\}$  and path  $Z = \{a = z_0, z_1, z_2, \ldots, z_{l_z-1}, z_{l_z} = b\}$ , where the edge-length of the paths, denoted by  $l_x$ ,  $l_y$ , and  $l_z$ , are larger or equal than 2. A graphic representation of  $H^{l_x,l_y,l_z}$  is shown in Figure 5.

**Lemma 4.** Any  $H^{l_x,l_y,l_z}$  graph such that  $l_z \geq l_y \geq l_x > 3$  does not belong to AND(1).

*Proof.* The proof is by contradiction. Let  $\mathcal{R}$  be an AND(1)-realization of  $H^{l_x,l_y,l_z}$ . By Lemma 1, Condition 1, the extreme vertices of the realization must be neighbors. Then, both extremes belong to the same path. Without loss of generality, we assume that vertex a is placed before than vertex b in the realization  $(a <_{\mathcal{R}} b)$  and that both extremes belong to path X, say  $x_k$  and  $x_{k+1}$  with

 $k \in \{0, \dots, l_x - 1\}$ . Therefore, according to Lemma 1, the induced cycles  $X \cup Y$  and  $X \cup Z$  must be oriented clockwise

$$(x_k,\ldots,x_0,y_1,\ldots,y_{l_n-1},b,\ldots,x_{l_n-1},\ldots,x_{k-1})$$

and anti-clockwise

$$(x_k,\ldots,x_0,z_1,\ldots,z_{l_v-1},b,\ldots,x_{l_x-1},\ldots,x_{k-1}),$$

respectively.

On the other hand, from Condition 1 of Lemma 1, it holds that a labeling l of an induced cycle satisfies the following property: if for two vertices u,v it holds l(u) < l(v-1) then  $\pi(u) < \pi(v)$ , i.e.,  $u <_{\mathcal{R}} v$ . Thus, in the cycle  $X \cup Y$ ,  $y_2 <_{\mathcal{R}}$  and  $a <_{\mathcal{R}} y_j$  for all j > 2. Symmetrically, for the cycle  $X \cup Z$ ,  $a <_{\mathcal{R}} z_j$  for all j > 2 and  $z_2 <_{\mathcal{R}} b$ .

Finally, in the induced realization of cycle  $Y \cup Z$ , the only possible pairs of extreme vertices are  $(a, y_1), (y_1, y_2)$  and  $(a, z_1), (z_1, z_2)$ . If the extremes are  $(a, y_1)$  or  $(y_1, y_2)$ , the cycle  $Y \cup Z$  is oriented anti-clockwise and  $b <_{\mathcal{R}} y_2$  which is a contradiction. Otherwise, if  $(a, z_1)$  or  $(z_1, z_2)$  are the extremes, cycle  $Y \cup Z$  is oriented clockwise and  $b <_{\mathcal{R}} z_2$  which is also a contradiction.

We shall see now that, indeed, the smaller cases  $H^{2,l_y,l_z}$  and  $H^{3,l_y,l_z}$  are minimal graphs that separate AND(1) from c-AND(1).

**Lemma 5.** Any  $H^{l_x,l_y,l_z}$  graph such that  $l_z \geq l_y \geq l_x > 2$  does not belong to c-AND(1).

The proof of this lemma follows the same ideas of the proof of Lemma 4.

Proof. The proof is by contradiction. Let  $\mathcal{R}$  be an c-AND(1)-realization of  $H^{l_x,l_y,l_z}$ . Paths X,Y and Z are defined as the previous proof. Since the extreme vertices of the realization must be neighbors (Lemma 1, Condition 1), then both extremes belong to the same path. W.l.o.g., we assume that vertex a is placed before than vertex b in the realization ( $a <_{\mathcal{R}} b$ ) and that both extremes belong to X, says  $x_k$  and  $x_{k+1}$  with  $k \in \{0, \ldots, l_x - 1\}$ . Therefore, according to Lemma 1, the induced cycles  $X \cup Y$  and  $X \cup Z$  must be oriented clockwise and anti-clockwise, respectively. That is:

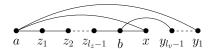
$$x_k, \ldots, x_0, y_1, \ldots, y_{l_y-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1}$$

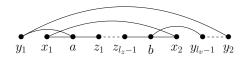
and

$$x_k, \ldots, x_0, z_1, \ldots, z_{l_v-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1},$$

respectively. Thus,  $y_1 <_{\mathcal{R}} b$  and  $z_1 <_{\mathcal{R}} b$ .

Consider the induced realization of cycle  $Y \cup Z$ . By the previous discussion, we conclude that a is the left extreme vertex of the induced realization. Thus, right extreme have to be  $y_1$  or  $z_1$ . Then, either  $b <_{\mathcal{R}} y_1$  or  $b <_{\mathcal{R}} z_1$  which both are contradictions.





**Fig. 6.** This figure shows a graphic representation of  $H^{2,l_y,l_z}$  and  $H^{3,l_y,l_z}$  where vertices are ordered according to the ordering given in the proof of Lemma 6.

With the previous Lemma we have presented an infinite family of graphs that do not belong c-AND(1). Nevertheless, some of these graphs do belong to AND(1).

**Lemma 6.** Graphs  $H^{2,l_y,l_z}$  and  $H^{3,l_y,l_z}$  belong to AND(1) for any  $l_y$  and  $l_z \geq 2$  and  $l_y$  and  $l_z \geq 3$ , respectively.

The proof of this lemma follows by giving orderings of the set of vertices of  $H^{2,l_y,l_z}$  and  $H^{3,l_y,l_z}$  that satisfy the four point condition for AND(1). Figure 6 shows graphically such orders.

*Proof.* Consider any  $H^{2,l_y,l_z}$  graph. In this case, the first path has length 2, therefore, we denote its vertex by x without subindex. In order to prove the Lemma, we give an ordering of the vertices of  $H^{2,l_y,l_z}$  that satisfies the four point condition for AND(1). Consider the following ordering for  $V(H^{2,l_y,l_z})$ :

$$a, z_1, z_2, \dots, z_{l_z-1}, b, x, y_{l_y-1}, y_{l_y-2}, \dots, y_1.$$

For the *i*th vertex, we define  $p_i$  to be equal to *i*. The intervals are defined as follow:  $I_a = [p_a, p_{y_1}]; I_{z_i} = [p_{z_i} - 1, p_{z_i} + 1]; I_b = [p_b, p_{l_y}]; I_x = [p_a, p_x]; I_{y_1} = [p_b, p_{y_1}]; I_{y_i} = [p_{y_i} - 1, p_{y_i} + 1]; I_{y_1} = [p_a, p_{y_1}].$ 

As it can be seen in Figure 6, according to this ordering of the vertices the only pair of edges that crosses one to another are edges ax and  $by_{l_y-1}$ . Since vertices b and x are connected by an edge in  $H^{2,l_y,l_z}$ , the four point condition is satisfied. Therefore, the graph  $H^{2,l_y,l_z}$  belongs to AND(1).

Consider any  $H^{3,l_y,l_z}$  graph, we give an ordering of  $V(H^{3,l_y,l_z})$  that satisfies the four point condition for AND(1). Consider the following order for  $V(H^{3,l_y,l_z})$ :

$$y_1, x_1, a, z_1, z_2, \dots, z_{l_z-1}, b, x_2, y_{l_y-1}, y_{l_y-2}, \dots, y_2.$$

For the *i*th vertex, we define  $p_i$  to be equal to *i*. The intervals are defined as follow:  $I_{y_1} = [p_{y_1}, p_{y_2}]; I_{x_1} = [p_{x_1}, p_{x_2}]; I_a = [p_{y_1}, p_a + 1]; I_{z_i} = [p_{z_i} - 1, p_{z_i} + 1]; I_b = [p_b - 1, p_{y_{l_y}}]; I_{y_i} = [p_{y_i} - 1, p_{y_i} + 1]; I_{y_2} = [p_{y_1}, p_{y_2}].$ 

In order to finish the proof, we have to check that this ordering satisfies the four point condition for AND(1). As it can be seen in Figure 6, there are two pair of edges that crosses one to each other. One pair is composed by edges  $y_1a$  and  $x_1x_2$ . Since vertices a and  $x_1$  are neighbors, the condition holds. The second pair is composed by edges  $x_1x_2$  and  $by_{l_y-1}$ . Since vertices  $x_2$  and b are neighbors, the condition holds. Therefore, any graph  $H^{3,l_y,l_z}$  belongs to AND(1).

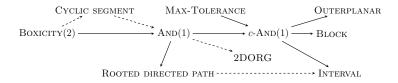


Fig. 7. Relation between the graph classes in the document. Arrows point from the superclass to the subclass. Dotted lines represent previous results and solid lines represent results proved in this document.

We consider important to stress the complete bipartite graph  $K_{2,3}$  as a particular case of Lemma 5 and Lemma 6, i.e.,  $K_{2,3}$  belongs to AND(1) but it does not belong to c-AND(1). Such an importance comes from the fact that  $K_{2,3}$  is the smallest complete bipartite graph that does not belong to c-AND(1). As a consequence of Lemma 5 and the fact that the property of belonging to c-AND(1) is hereditary, we can say that any graph that contains a  $H^{l_x,l_y,l_z}$  as an induced subgraph does not belong to c-AND(1). On the other hand, from Lemma 6 we know that some of these graphs do belong to AND(1).

## 7 Future work

Our results are graphical expressed in Figure 7. On the other hand, our work suggests several directions for future research. In our opinion, the most natural question is to find a combinatorial characterization for the c-AND(1) family. Another interesting open problem concerns to determine the complexity of the recognition problem for both AND(1) and c-AND(1) families. The study of higher dimensions of the families is an alternative way to continue this research. Another interesting question is the study of the family of graphs generated when points are embedded in a different metric space, for instance the d-dimensional torus.

Acknowledgements The authors would like to thank Antonio Fernández Anta and Marcos Kiwi because they are strongly involved in the origins of this study, even more, they contributed with enlightening talks and ideas.

#### References

- Bil92. T. Bilski. Embedding graphs in books: A survey. *IEE Proceedings-E*, 139(2):134–138, March 1992.
- BK79. F. Bernhart and P. C. Kainen. The book thickness of a graph. *Journal of Combinatorial Theory, Series B*, 27(3):320 331, 1979.
- BLS99. A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes: A Survey*. Siam Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, 1999.
- BM07. J.J.A. Bondy and U.S.R. Murty. *Graph Theory*. Graduate Texts in Mathematics Series. Springer London, 2007.

- CCH<sup>+</sup>. D. Cantanzaro, S. Chaplick, B. Halldórsson, M. Halldórsson, and J. Stacho. Max point-tolerance graphs.
- EGP64. P. Erdös, A. W. Goodman, and L. Posa. The representation of a graph by set intersections, 1964.
- FCFM09. M. Farach-Colton, R. J. Fernandes, and M. A. Mosteiro. Bootstrapping a hop-optimal network in the weak sensor model. *ACM Transactions on Algorithms*, 5(4), 2009.
- Feu. L. Feuilloley. Personal communication.
- GM82. M. C. Golumbic and C. L. Monma. A generalization of interval graphs with tolerances. *Combinatorica*, 1982.
- Gol04. M. C. Golumbic. Algorithmic Graph Theory and Perfect Graphs. Annals of Discrete Mathematics. Elsevier, 2004.
- GT04. M. C. Golumbic and A. N. Trenk. *Tolerance Graphs*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.
- Gyö84. E. Györi. A minimax theorem on intervals. Journal of Combinatorial Theory, Series B, 37(1):1–9, 1984.
- Hix13. T. S. Hixon. Hook graphs and more: Some contributions to geometric graph theory. Master's thesis, Technische Universität Berlin, 2013.
- HK01. P. Hlineny and J. Kratochvíl. Representing graphs by disks and balls (a survey of recognition-complexity results). Discrete Mathematics, 229(1 3):101 124, 2001.
- KKLS06. M. Kaufmann, J. Kratochvíl, K. A. Lehmann, and A. R. Subramanian. Maxtolerance graphs as intersection graphs: cliques, cycles, and recognition. In Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm, SODA '06, pages 832–841, New York, NY, USA, 2006. ACM.
- Koe36. P. Koebe. Kontaktprobleme der konformen abbildung. Berichte über die Verhand-lungen de Sächsischen, Akad. Wiss. Leipzig, Math.-Phys., 88:141– 164, 1936.
- KWZ08. F. Kuhn, R. Wattenhofer, and A. Zollinger. Ad hoc networks beyond unit disk graphs. Wireless Networks, 14:715–729, 2008.
- Lub91. A. Lubiw. A weighted min-max relation for intervals. Journal of Combinatorial Theory, Series B, 53(2):151–172, 1991.
- MM99. T.A. McKee and F.R. McMorris. *Topics in Intersection Graph Theory*. Siam Monographs on Discrete Mathematics and Applications. Siam, 1999.
- Ola91. S. Olariu. An optimal greedy heuristic to color interval graphs. *Information Processing Letters*, 37(1):21 25, 1991.
- Rob69a. F. S. Roberts. Indifference graphs. In F. Harary, editor, *Proof Techniques in Graph Theory*, pages 139–146. Academic Press, New York, 1969.
- Rob69b. F. S. Roberts. On the boxicity and cubicity of a graph. In W. T. Tutte, editor, *Recent progress in combinatorics*, pages 301–310. Academic press, New York, 1969.
- SMS45. S.-M. and E. Sur. Sur deux propriétés des classes d'ensembles. (french). Fundamenta Mathematicae, 33(1):303–307, 1945.
- Spi03. J.P. Spinrad. Efficient Graph Representations.: The Fields Institute for Research in Mathematical Sciences. Fields Institute Monographs, 19. American Mathematical Society, 2003.
- Tuc72. A. Tucker. A structure theorem for the consecutive 1's property. *Journal of Combinatorial Theory, Series B*, 12(2):153–162, 1972.
- WG86. M. Waterman and J. Griggs. Interval graphs and maps of dna. *Bulletin of Mathematical Biology*, 48:189–195, 1986.