Investigation of the Kolmogorov–Wiener filter for treatment of fractal processes on the basis of the Chebyshev polynomials of the second kind

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Abstract. We consider the Kolmorogov–Wiener filter for continuous fractal processes with a power-law structure function. The corresponding filter is used for data forecast; the noiseless case is considered. The aim of the paper is to obtain the weight function for the corresponding filter based on the integral Wiener–Hopf equation. The problem under consideration is important, for example, for traffic forecast in telecommunication systems and for the forecast of the chemical composition of cast iron. An exact analytical solution for the corresponding equation meets difficulties, so an approximate solution is sought in the form of a truncated Chebyshev polynomial expansion. The Chebyshev polynomials of the second kind are used. The behavior of the polynomial solutions for different numbers of polynomials is investigated. The results are compared with the corresponding results of our previous paper where another polynomial set is used. It is found that the corresponding behavior is almost identical for different polynomial sets.

Keywords: Kolmogorov–Wiener filter weight function, continuous stationary fractal processes, power-law structure function, Chebyshev polynomials of the second kind.

1 Introduction

We consider the Kolmogorov–Wiener filter for data forecast for continuous fractal processes. Nowadays fractal processes take place in a huge variety of different systems, see, for example, [1–5] and various references in [5].

The problem of the Kolmogorov–Wiener weight function search for continuous fractal processes with a power-law structure function is stated in [4]. In that paper it is mentioned that such a model could be suitable for teletraffic description in IEEE 802.11b networks and for the routers between the internal networks and the Internet. In fact, the Wiener-Hopf integral equation is a Fredholm integral equation of the first kind. In [4] a simplified Volterra integral equation is used instead of the Fredholm one and the idea of the solution for the Volterra integral equation is described. Finally, an exact analytical solution for the corresponding equation was obtained in [6].

Maybe, in some simplified cases the Volterra integral equation can indeed be applied to the investigation of data forecast in real systems. But in the general case it is not applicable, and the Fredholm integral equation should be solved instead of the Volterra one. In contrast to the Volterra integral equation, an exact analytical solution for a Fredholm integral equation meets difficulties. Thus, an approximate solution for the corresponding equation is sought. The method of a truncated orthogonal polynomial expansion is rather popular in the literature in order to obtain an approximate solution for the Fredholm integral equation of the first kind, see, for example, the corresponding investigation in the framework of statistical physics [7–10].

In paper [11] such a method was applied to the problem under consideration. A set of polynomials which are orthogonal without weight is used in [11]. It is shown that although the method can give reliable results in a rather wide range of parameters, it has some drawbacks in the case of a power-law structure function. The most significant drawback is the fact that the accuracy of the method does not necessarily increase with the number of polynomials. For some numbers of polynomials the method gives reliable results, but for other numbers it may fail. Most likely the reason is that the corresponding correlation function, which is the kernel of the Wiener–Hopf integral equation, is not a positively defined function, so the convergence of the method is not guaranteed, see a similar discussion in the framework of statistical physics in [12].

But, anyway, the question arises: may the results be better if we use another polynomial set? Is the behavior of the polynomial solutions identical for different sets of polynomials? This interesting question should be investigated because it is rather hard to propose another analytical method for the solution for the corresponding Wiener–Hopf equation. In this paper we use a set of the Chebyshev polynomials of the second kind. So, the aim of this work is to obtain the Kolmogorov–Wiener filter weight function on the basis of a truncated expansion in the Chebyshev polynomials of the second kind and to compare the results with the results of paper [11].

2 Description of the truncated polynomial expansion method

We consider stationary continuous fractal processes with a power-law structure function. The correlation function of such processes has the form [4]

$$R(t) = \sigma^2 - \frac{\alpha}{2}|t|^{2H} \tag{1}$$

where σ is the process variance, H is the Hurst exponent and α is a constant.

Let the filter input signal be defined for $t \in [0, T]$. As is known [13], in such a case the Kolmogorov–Wiener filter weight function $h(t)$ is the solution of the following Wiener–Hopf integral equation

$$\int_0^T d\tau h(\tau) R(t-\tau) = R(t+k) \tag{2}$$

where $k \ll T$ is the time interval for which the forecast is made. Such an equation can hardly be solved exactly, so an approximate solution should be found.

In paper [11] a truncated polynomial expansion method is used, and the following polynomials are taken:

$$S_n(\tau) = \frac{S_n'(\tau)}{\sqrt{\int_0^T dt \left(S_n'(t)\right)^2}} \tag{3}$$

where

$$S_n'(\tau) = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & \tau & \tau^2 & \cdots & \tau^n \end{vmatrix}, \quad \mu_n = \int_0^T x^n dx = \frac{T^{n+1}}{n+1}. \tag{4}$$

Such polynomials are orthogonal without weigh on $t \in [0,T]$:

$$\int_0^T dt S_n(t) S_m(t) = \delta_{mn} \tag{5}$$

where δ_{mn} is the Kronecker delta.

In this paper we take another polynomial set. We use the Chebyshev polynomials of the second kind. Their explicit expressions are [14]

$$U_n(x) = \sum_{k=0}^{[n/2]} C_{n+1}^{2k+1} x^{n-2k} \left(x^2-1\right)^k \tag{6}$$

where $[n/2]$ is the integer part of $n/2$. They are orthogonal on $x \in [-1,1]$ with the orthogonality condition [14]:

$$\int_{-1}^1 U_n(x) U_m(x) \sqrt{1-x^2}\, dx = \frac{\pi}{2}\delta_{mn} = \begin{cases} \pi/2, m = n \\ 0, m \neq n \end{cases} \tag{7}$$

But we need a polynomial set that is orthogonal on $t \in [0,T]$. On the basis of (6) after making the following change of the variables:

$$z = x+1, \; y = zT/2 \tag{8}$$

one can derive the following expression:

$$\int_0^T U_n\left(\frac{2y}{T}-1\right)U_m\left(\frac{2y}{T}-1\right)\sqrt{1-\left(\frac{2y}{T}-1\right)^2}\,dy=\frac{T\pi}{4}\delta_{mn}\,. \tag{9}$$

So the polynomials $U_n\left(\frac{2y}{T}-1\right)$ are orthogonal on $y\in[0,T]$ with the weight $\sqrt{1-\left(\frac{2y}{T}-1\right)^2}$. So, an approximate solution of the integral equation (2) is sought in the form

$$h(\tau)=\sum_{s\geq 0} g_s U_s\left(\frac{2\tau}{T}-1\right). \tag{10}$$

After substitution of (10) into (2), one can obtain

$$\sum_{s\geq 0} g_s\int_0^T d\tau U_n\left(\frac{2\tau}{T}-1\right)R(t-\tau)=R(t+k) \tag{11}$$

which after multiplying by $U_n\left(\frac{2t}{T}-1\right)$ and integrating over t leads to

$$\sum_{s\geq 0} g_s\int_0^T\int_0^T dt d\tau U_n\left(\frac{2t}{T}-1\right)U_s\left(\frac{2\tau}{T}-1\right)R(t-\tau)=\int_0^T dt U_n\left(\frac{2t}{T}-1\right)R(t+k)\,. \tag{12}$$

Denoting

$$G_{ns}=\int_0^T\int_0^T dt d\tau U_n\left(\frac{2t}{T}-1\right)U_s\left(\frac{2\tau}{T}-1\right)R(t-\tau)\,,\ b_n=\int_0^T dt U_n\left(\frac{2t}{T}-1\right)R(t+k) \tag{13}$$

one can rewrite (12) as

$$\sum_{s\geq 0} g_s G_{ns}=b_n\,,\ n\geq 0\,. \tag{14}$$

As can be seen, (14) is an infinite set of linear equations in the unknown coefficients g_s. This set can hardly be treated, so it should be artificially truncated to a finite number of equations:

$$\sum_{s=0}^{l-1} g_s G_{ns}=b_n\,,\quad n=\overline{0,l-1}\,. \tag{15}$$

The Kolmogorov–Wiener filter weight function

$$h(\tau)=\sum_{s=0}^{l-1} g_s U_s\left(\frac{2\tau}{T}-1\right) \quad (16)$$

where the coefficients g_s are the solutions of (15) is the weight function in the l-polynomial approximation.

Here and in what follows the quantities G_{ns} are called the integral brackets. On the basis of (13) after making the following change of the variables:

$$x=\frac{2\tau}{T}-1,\ y=\frac{2t}{T}-1 \quad (17)$$

one can obtain the following expression for the integral brackets:

$$G_{ns}=\frac{T^2}{4}\int_{-1}^{1}\int_{-1}^{1} dxdy U_n(x)U_s(y)R\left(\frac{T}{2}y-\frac{T}{2}x\right). \quad (18)$$

It should be stressed that such a choice of polynomials is rather convenient. As can be seen from (6), the polynomials $U_n(x)$ obey the property

$$U_n(x)=\begin{cases}U_n(-x), n\vdots 2\\ -U_n(-x), n\not\vdots 2\end{cases}. \quad (19)$$

By changing x to $-x$ and y to $-y$ in (18), on the basis of (19) it can be seen that $G_{ns}=0$ if n and s are of different parity. This property takes place because the correlation function (1) is an even function. Also, the evenness of the correlation function leads to the fact that $G_{ns}=G_{sn}$. These two properties allow one to calculate G_{ns} by a straightforward calculation only for $n\geq s$ and n, s being of the same parity. Such a fact significantly reduces the computing time.

In the following section the numerical behavior of the l-polynomial approximation solutions is investigated.

3 Behavior of polynomial solutions

The behavior of the polynomial solutions is investigated for the parameters

$$T=100,\ k=3,\ \sigma=1.2,\ H=0.8,\ \alpha=3\cdot 10^{-3}. \quad (20)$$

First of all, this set does not contradict the inequality $|R(t)|\leq R(0)$. Secondly, this set is investigated in [11]. For the set (20) the numerical values for the coefficients in (16) in the l–polynomial approximations are given in Table 1.

Table 1. Numerical values for the coefficients of polynomials in (16) for parameters (20)

l	Coefficients g_0, g_1, …, g_{l-1} rounded off to three significant digits
1	$4.86\cdot10^{-3}$.
2	$4.86\cdot10^{-3}$, $-1.46\cdot10^{-2}$.
3	$1.50\cdot10^{-2}$, $-1.46\cdot10^{-2}$, $-1.03\cdot10^{-1}$.
4	$1.50\cdot10^{-2}$, $-8.84\cdot10^{-3}$, $-1.03\cdot10^{-1}$, $-1.62\cdot10^{-2}$.
5	$-1.96\cdot10^{-3}$, $-8.84\cdot10^{-3}$, $2.53\cdot10^{-2}$, $-1.62\cdot10^{-2}$, $4.28\cdot10^{-2}$.
6	$-1.96\cdot10^{-3}$, $-5.84\cdot10^{-3}$, $2.53\cdot10^{-2}$, $-1.37\cdot10^{-2}$, $4.28\cdot10^{-2}$, $-1.64\cdot10^{-2}$.
7	$-1.41\cdot10^{-3}$, $-5.84\cdot10^{-3}$, $1.22\cdot10^{-2}$, $-1.37\cdot10^{-2}$, $2.71\cdot10^{-2}$, $-1.64\cdot10^{-2}$, $3.05\cdot10^{-2}$.
8	$-1.41\cdot10^{-3}$, $-3.97\cdot10^{-3}$, $1.22\cdot10^{-2}$, $-1.06\cdot10^{-2}$, $2.71\cdot10^{-2}$, $-1.66\cdot10^{-2}$, $3.05\cdot10^{-2}$, $-1.64\cdot10^{-2}$.
9	$-1.61\cdot10^{-3}$, $-3.97\cdot10^{-3}$, $7.26\cdot10^{-3}$, $-1.06\cdot10^{-2}$, $1.92\cdot10^{-2}$, $-1.66\cdot10^{-2}$, $2.84\cdot10^{-2}$, $-1.64\cdot10^{-2}$, $2.69\cdot10^{-2}$.
10	$-1.61\cdot10^{-3}$, $-2.68\cdot10^{-3}$, $7.26\cdot10^{-3}$, $-8.16\cdot10^{-3}$, $1.92\cdot10^{-2}$, $-1.44\cdot10^{-2}$, $2.84\cdot10^{-2}$, $-1.85\cdot10^{-2}$, $2.69\cdot10^{-2}$, $-1.63\cdot10^{-2}$.
11	$-1.90\cdot10^{-3}$, $-2.68\cdot10^{-3}$, $4.40\cdot10^{-3}$, $-8.16\cdot10^{-3}$, $1.42\cdot10^{-2}$, $-1.44\cdot10^{-2}$, $2.40\cdot10^{-2}$, $-1.85\cdot10^{-2}$, $2.95\cdot10^{-2}$, $-1.63\cdot10^{-2}$, $2.51\cdot10^{-2}$.
12	$-1.90\cdot10^{-3}$, $-1.70\cdot10^{-3}$, $4.40\cdot10^{-3}$, $-6.19\cdot10^{-3}$, $1.42\cdot10^{-2}$, $-1.21\cdot10^{-2}$, $2.40\cdot10^{-2}$, $-1.75\cdot10^{-2}$, $2.95\cdot10^{-2}$, $-1.99\cdot10^{-2}$, $2.51\cdot10^{-2}$, $-1.63\cdot10^{-2}$.
13	$-2.19\cdot10^{-3}$, $-1.70\cdot10^{-3}$, $2.41\cdot10^{-3}$, $-6.19\cdot10^{-3}$, $1.06\cdot10^{-2}$, $-1.21\cdot10^{-2}$, $1.98\cdot10^{-2}$, $-1.75\cdot10^{-2}$, $2.74\cdot10^{-2}$, $-1.99\cdot10^{-2}$, $3.03\cdot10^{-2}$, $-1.63\cdot10^{-2}$, $2.41\cdot10^{-2}$.
14	$-2.19\cdot10^{-3}$, $-9.32\cdot10^{-4}$, $2.41\cdot10^{-3}$, $-4.59\cdot10^{-3}$, $1.06\cdot10^{-2}$, $-9.94\cdot10^{-3}$, $1.98\cdot10^{-2}$, $-1.56\cdot10^{-2}$, $2.74\cdot10^{-2}$, $-1.99\cdot10^{-2}$, $3.03\cdot10^{-2}$, $-2.10\cdot10^{-2}$, $2.41\cdot10^{-2}$, $-1.62\cdot10^{-2}$.
15	$-2.46\cdot10^{-3}$, $-9.32\cdot10^{-4}$, $9.03\cdot10^{-4}$, $-4.59\cdot10^{-3}$, $7.78\cdot10^{-3}$, $-9.94\cdot10^{-3}$, $1.62\cdot10^{-2}$, $-1.56\cdot10^{-2}$, $2.44\cdot10^{-2}$, $-1.99\cdot10^{-2}$, $3.01\cdot10^{-2}$, $-2.10\cdot10^{-2}$, $3.10\cdot10^{-2}$, $-1.62\cdot10^{-2}$, $2.34\cdot10^{-2}$.
16	$-2.46\cdot10^{-3}$, $-3.04\cdot10^{-4}$, $9.03\cdot10^{-4}$, $-3.27\cdot10^{-3}$, $7.78\cdot10^{-3}$, $-8.05\cdot10^{-3}$, $1.62\cdot10^{-2}$, $-1.36\cdot10^{-2}$, $2.44\cdot10^{-2}$, $-1.86\cdot10^{-2}$, $3.01\cdot10^{-2}$, $-2.19\cdot10^{-2}$, $3.10\cdot10^{-2}$, $-2.18\cdot10^{-2}$, $2.34\cdot10^{-2}$, $-1.61\cdot10^{-2}$.
17	$-2.70\cdot10^{-3}$, $-3.04\cdot10^{-4}$, $-3.05\cdot10^{-4}$, $-3.27\cdot10^{-3}$, $5.54\cdot10^{-3}$, $-8.05\cdot10^{-3}$, $1.32\cdot10^{-2}$, $-1.36\cdot10^{-2}$, $2.13\cdot10^{-2}$, $-1.86\cdot10^{-2}$, $2.82\cdot10^{-2}$, $-2.19\cdot10^{-2}$, $3.23\cdot10^{-2}$, $-2.18\cdot10^{-2}$, $3.15\cdot10^{-2}$, $-1.61\cdot10^{-2}$, $2.30\cdot10^{-2}$.
18	$-2.70\cdot10^{-3}$, $2.22\cdot10^{-4}$, $-3.05\cdot10^{-4}$, $-2.15\cdot10^{-3}$, $5.54\cdot10^{-3}$, $-6.41\cdot10^{-3}$, $1.32\cdot10^{-2}$, $-1.16\cdot10^{-2}$, $2.13\cdot10^{-2}$, $-1.69\cdot10^{-2}$, $2.82\cdot10^{-2}$, $-2.12\cdot10^{-2}$, $3.23\cdot10^{-2}$, $-2.35\cdot10^{-2}$, $3.15\cdot10^{-2}$, $-2.24\cdot10^{-2}$, $2.30\cdot10^{-2}$, $-1.61\cdot10^{-2}$.

The investigation is made up to the 18-polynomial approximation; the Wolfram Mathematica 11.0 package is used. The obtained weight function in each approximation is substituted into the integral equation (2) and the left-hand and the right-hand sides of the equation are numerically compared.

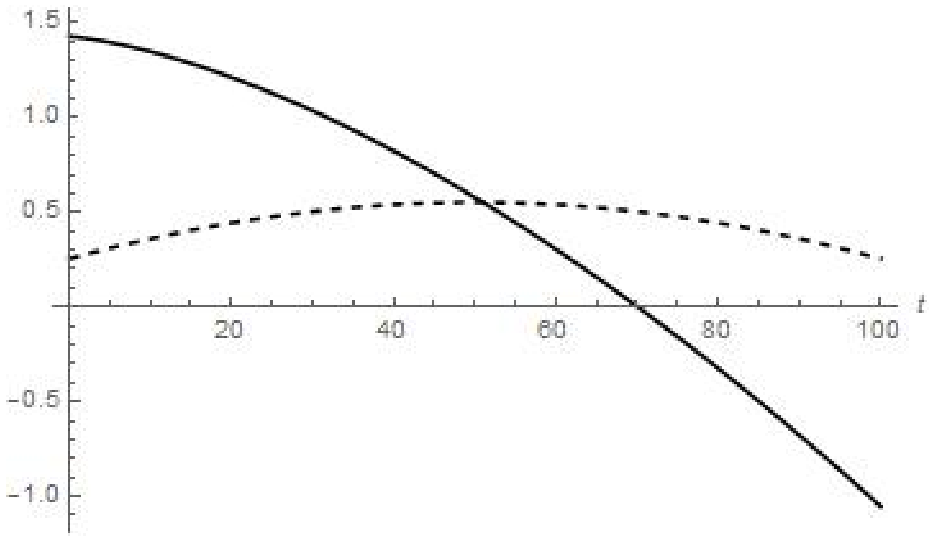

Fig. 1. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the one-polynomial approximation.

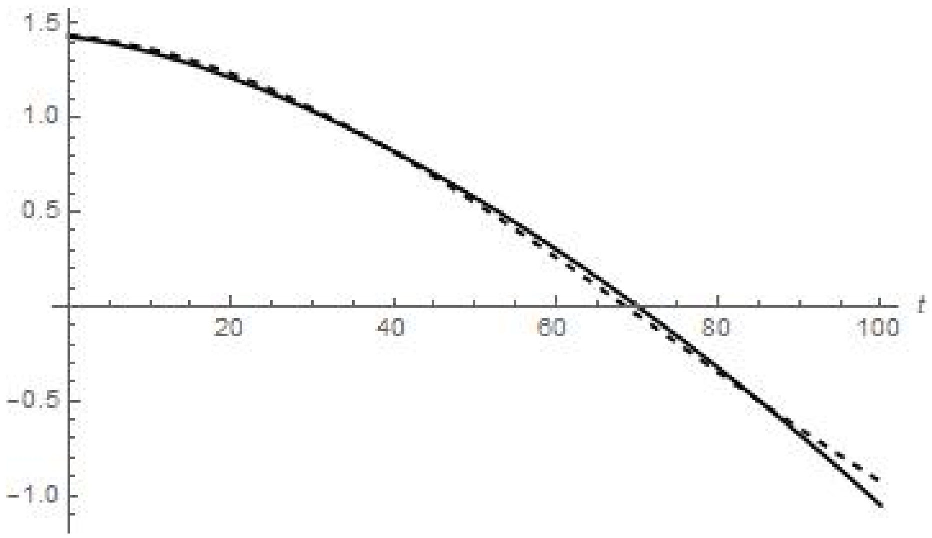

Fig. 2. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the two-polynomial approximation.

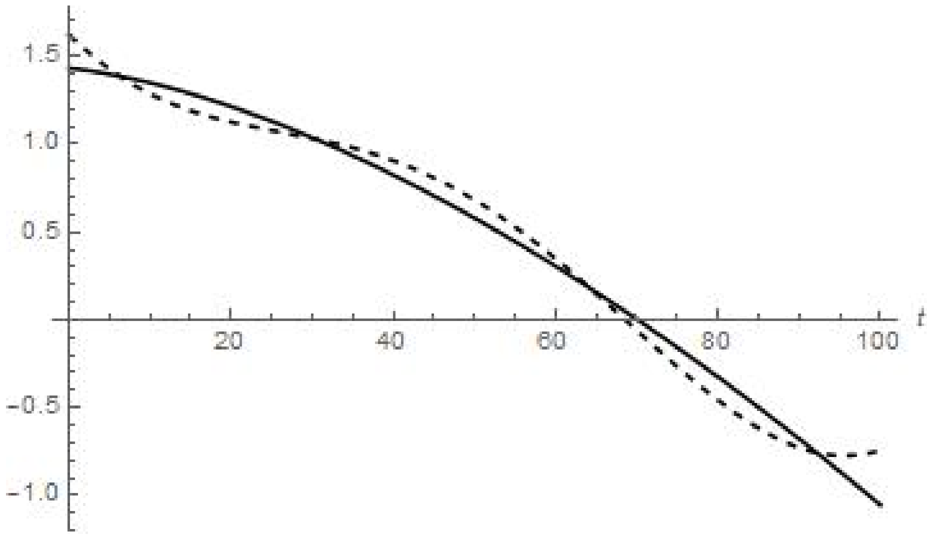

Fig. 3. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the three-polynomial approximation.

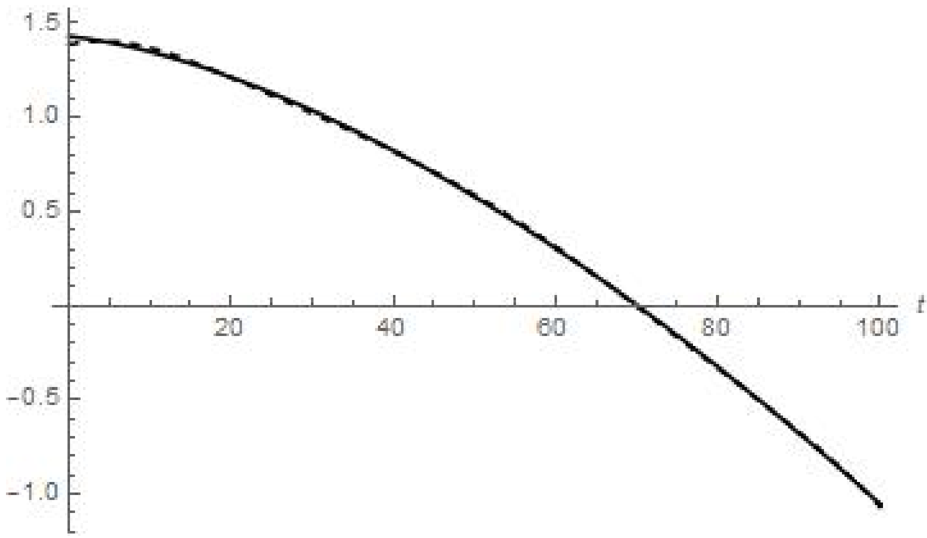

Fig. 4. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the five-polynomial approximation.

As can be seen from Fig.1 – Fig.3, the one-polynomial approximation is not accurate, but the two-polynomial approximation is rather accurate. The corresponding graph for the four-polynomial approximation in not given because it is almost identical to the graph for the three-polynomial one. The four- and three-polynomial approximations are worse than the two-polynomial one, but better than the one-polynomial one. The five-polynomial approximation is accurate (see Fig. 4). The accuracy slowly increases with the number of polynomials from the five- to the eight-polynomial approximations.

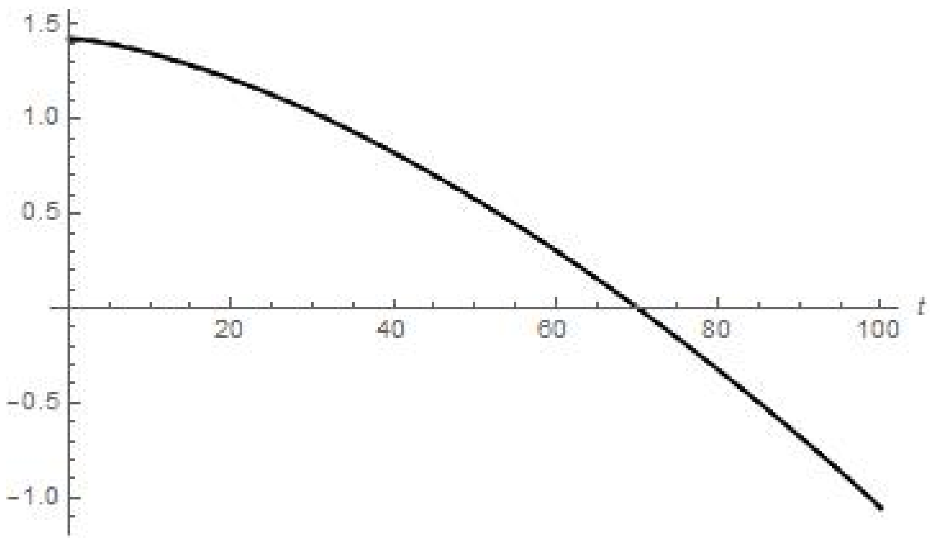

Fig. 5. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the eight-polynomial approximation.

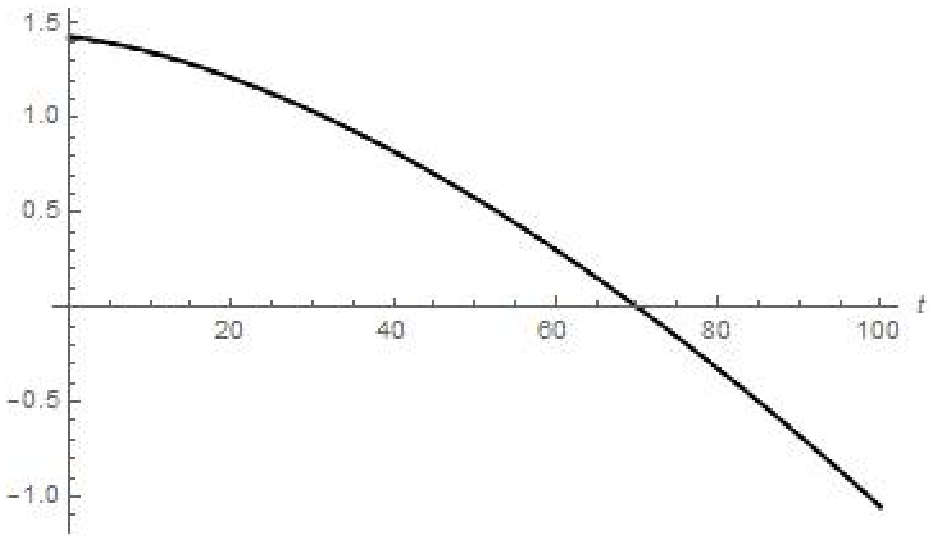

Fig. 6. Comparison of the left-hand and right-hand sides of eq. (2) for parameters (20) for the eighteen-polynomial approximation.

As can be seen from Fig. 5, the eight-polynomial approximation gives an almost ideal coincidence between the left-hand and the right-hand sides of the integral equation (2). But the approximations for the numbers of polynomials from nine to fifteen fail. For these approximations the curves for the left-hand and right-hand sides of eq. (2) are very far from each other. But the sixteen-, seventeen- and eighteen-polynomial approximations again give almost ideal results. The graphs for them are in fact identical and the graph for the eighteen-polynomial approximation is given in Fig. 6.

Such behavior of polynomial solutions is rather strange, but is can be explained as follows. The kernel of the integral equation (2) is not a positively defined function, so the convergence of the polynomial procedure is not guaranteed. In other words, the accuracy of the method does not necessarily increase with the number of polynomials.

It should be stressed that the behavior of the polynomial solutions described in [11] for the polynomial set (3) is, in fact, the same. The behavior of polynomial solutions is also investigated for the sets of parameters $T=10$, $k=3$, $\sigma=1.2$, $H=0.8$, $\alpha=10^{-1}$ and $T=1000$, $k=3$, $\sigma=1.2$, $H=0.8$, $\alpha=8\cdot10^{-5}$. For the corresponding sets of parameters the behavior of the polynomial solutions for the polynomial sets (3) and (10) is almost identical. In [11] it is stressed that although the accuracy of the polynomial solutions may not increase with the number of polynomials and some approximations may fail, in a rather wide range of parameters (from $T=10$ to $T=1000$) some of the approximations give reliable results.

4 Conclusions

The Kolmogorov–Wiener filter weight function is investigated for continuous fractal processes with a power-law structure function. The method of a truncated orthogonal polynomial expansion is used in order to obtain an approximate solution of the corresponding Wiener–Hopf integral equation. In this paper the Chebyshev polynomials of the second kind which are orthogonal with weight on $t\in[0,T]$ are used. The numerical calculations are made on the basis of the Wolfram Mathematica 11.0 package.

It is found that the behavior of the polynomial approximations for the Chebyshev polynomials (10) and the behavior of the corresponding approximations for the polynomials (3), which is investigated in [11], are in fact the same. So, it may be concluded that the behavior of the polynomial solutions for the problem under consideration almost does not depend on the chosen polynomial set.

The proposed method of the approximate solution of the integral Wiener–Hopf equation for processes with a power-law structure functions has some drawbacks. The accuracy of the polynomial approximations may not increase with the number of polynomials, and some approximations may fail. This may happen because in such a case the kernel of the corresponding integral equation is not a positively defined function.

Nevertheless, in a rather wide range of parameters some polynomial approximations may give reliable results. Each approximation should be checked numerically before its further application to the investigation of data forecast in different systems.

References

1. Gusev, O., Kornienko, V., Gerasina, O., Aleksieiev O.: Fractal analysis for forecasting chemical composition of cast iron. In book “Energy Efficiency Improvement of Geotechnical Systems”, Taylor & Francis Group, London, 225–231 (2016).
2. Korniienko, V., Matsyuк, S., Udovik. I., Alekseyev, A.: Method and algorithms of nonlinear dynamic processes identification. Scientific Bulletin of National Mining University, No. 1, 98–103 (2016).
3. Kornienko, V., Gerasina, A., Gusev, A.: Methods and principles of control over the complex objects of mining and metallurgical production. In book “Energy Efficiency Improvement of Geotechnical Systems”, Taylor & Francis Group, London, 183-192 (2013).
4. Bagmanov, V. Kh., Komissarov, A. M., Sultanov, A. Kh.: Teletraffic forecast on the basis of fractal fliters. Bulletin of Ufa State Aviation Technical University, Vol. 9, No. 6 (24), 217–222 (2007), in Russian.
5. Pipiras V., Taqqu, M.: Long-Range Dependence and Self-Similarity. Cambridge University Press (2017).
6. Gorev, V. N., Gusev, A. Yu., Korniienko, V. I.: On the analytical solution of a Volterra integral equation for investigation of fractal processes. Radio Electronics, Computer Science, Control, No. 4, 42–50 (2018)
7. Gorev, V. N., Sokolovsky, A. I.: Plasma kinetic coefficients with account for relaxation processes. International Journal of Modern Physics B, Vol. 29, No. 32, 1550233 (2015).
8. Gorev, V. N., Sokolovsky, A. I.: Hydrodynamic, kinetic modes of plasma and relaxation damping of plasma oscillations. Ukrainian Journal of Physics, Vol. 60, No. 3, 232–246 (2015).
9. Loyalka, S. K., Tipton, E. L., Tompson, R.V.: Chapman–Enskog solutions to arbitrary order in Sonine polynomials I: Simple, rigid-sphere gas. Physica A, Vol. 379, 417–435 (2007).
10. Loyalka, S. K., Tipton, E. L., Tompson, R.V.: Chapman–Enskog solutions to arbitrary order in Sonine polynomials II: Viscosity in a binary, rigid-sphere, gas mixture. European Journal of Mechanics B/Fluids. Vol. 28, 335–352 (2009).
11. Gorev, V. N., Gusev, A. Yu., Korniienko, V. I.: Polynomial solutions for the Kolmogorov-Wiener filter weight function for fractal processes. Radio Electronics, Computer Science, Control, No. 2 (2019), in press.
12. Ziman, J. M.: Electrons and Phonons. The Theory of Transport Phenomena in Solids. Oxford University Press (2001).
13. Miller, S., Childers, D.: Probability and Random Processes With Applications to Signal Processing and Communications. Second edition. Amsterdam, Elseiver/Academic Press (2012).
14. Gradshteyn, I. S., Ryzhik, I. M.: Table of Integrals, Series, and Products. Eighth edition. Edited by Daniel Zwillinger and Victor Moll. Amsterdam, Elsevier/Academic Press, (2015)