Fractional Gaussian Noise Traffic Prediction Based on the Walsh Functions

Vyacheslav Gorev[a], Alexander Gusev[a] and Valerii Korniienko[a]

[a] *Dnipro University of Technology, 19 Dmytra Yavornytskoho ave., 49005 Dnipro , Ukraine*

Abstract

We investigate the theoretical fundamentals of the Kolmogorov–Wiener filter construction for the continuous telecommunication traffic prediction. The traffic is treated as a continuous fractional Gaussian noise. The integral equation for the filter weight function is solved with the help of the Galerkin method in the framework of which the unknown function is sought as a truncated series in orthogonal functions. The investigation is different to our previous papers based on the polynomial functions, in this paper we propose to realize the Galerkin method on the basis of truncated Walsh function expansion, which is the method enhancement. Such an enhancement is based on the idea that the Walsh functions are step ones, which allows one to obtain the analytical expressions for the integral brackets. So, the corresponding numerical calculation of the double integrals is not needed. Moreover, the proposed approach does not require the calculation of the products of very large and very small numbers. So, the proposed enhancement allows one to investigate a wider range of parameters and higher numbers of functions in contrast to the polynomial realizations. The approach developed in the paper may be applied to the practical telecommunication traffic prediction.

Keywords

Kolmogorov–Wiener filter, Galerkin method, continuous fractional Gaussian noise, Walsh functions, telecommunication traffic prediction.

1. Introduction and related works

In this paper we deal with the telecommunication traffic prediction. This problem is important for telecommunications and has many applications (see [1] and references therein). In particular, it is urgent for information security because security attacks may be detected if the traffic behavior significantly differs from the predicted one [1-2].

In telecommunication systems with packet transfer of data, the telecommunication traffic is treated as a self-similar process. In a rather simple model the telecommunication traffic is treated as fractional Gaussian noise. The practical applicability of the corresponding model is a debatable question [3]; however, fractional Gaussian noise is still used for the simulation of traffic data (see, for example, [4]). If the amount of data is rather large, the traffic may be treated as a continuous process [5], so in this paper we consider traffic as continuous fractional Gaussian noise.

Fractional Gaussian noise is a stationary random process [4], and the Kolmogorov–Wiener filter is applicable to the prediction of stationary processes [6]. This filter is linear and stationary (rather simple one), so it is logical enough to apply this filter to the prediction of stationary telecommunication traffic. There exist many different and rather complicated approaches to telecommunication traffic prediction (their brief overview is given in [1]); however, the approach based on the Kolmogorov–Wiener filter is not sufficiently developed in the literature.

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EMAIL: lordjainor@gmail.com (V. Gorev); gusev1950@ukr.net (A. Gusev); vikor7@ukr.net (V. Korniienko)
ORCID: 0000-0002-9528-9497 (V. Gorev); 0000-0002-0548-728X (A. Gusev); 0000-0002-0800-3359 (V. Korniienko)

The Kolmogorov-Wiener filter weigh function obeys the Wiener–Hopf integral equation, which is a Fredholm integral equation of the first kind [6]. An exact solution for this equation can hardly be treated analytically, and it is reasonable enough to investigate an approximate solution. Such an investigation may be realized with the help of the Galerkin method [7]. In the framework of the corresponding method, the solution is treated as a truncated orthogonal function series.

In our previous paper [8], we investigated the corresponding solution with the help of the polynomial functions. The use of polynomial expansions is rather popular nowadays in different fields of knowledge (see the applications to the solution of kinetic equations [9, 10]). It is shown [8] that the polynomial solutions give a good agreement of both sides of the integral equation if the number of polynomials is rather large. However, in our opinion, the polynomial expansion has some drawbacks. First of all, the analytical expressions for the so-called integral brackets are too cumbersome, and they may not be applicable if the number of polynomials is rather large. Moreover, the use of polynomials may lead to the product of very large and very small numbers, which may not be adequately treated numerically, that is why the number of polynomials that may be investigated and the range of traffic parameters may be limited. It should also be indicated that for other traffic models some polynomial approximations may fail even for a rather large number of polynomials (see the description for a power-law structure function model [11]).

In order to overcome the above-mentioned drawbacks, we use a truncated Walsh function expansion instead of a polynomial one. The Walsh functions are step ones [12], which allows one to derive the integral brackets analytically and to avoid the product of very large and very small numbers. The goal of the paper is to derive the Kolmogorov–Wiener filter weight function for the continuous telecommunication traffic prediction with the help of a truncated Walsh function expansion and to compare the both sides of the corresponding integral equation for the obtained weight function.

This paper is structured as follows. In Sec. 1 an introduction is given, in Sec. 2 the Wiener–Hopf integral equation, the Galerkin method and the Walsh functions are described, Sec. 3 contains the derivation of an algorithm for obtaining the above-mentioned weight function, Sec. 4 contains a numerical comparison of both sides of the Wiener-Hopf integral equation for the obtained weight function, in Sec. 5 conclusions are formulated, and Sec. 6 contains the refrences.

2. Wiener-Hopf integral equation, Galerkin method and Walsh functions

The unknown Kolmogorov–Wiener filter weight function $h(\tau)$ for the continuous telecommunication traffic prediction in the fractional Gaussian noise model obeys the following integral equation [8]:

$$\int_0^T d\tau h(\tau)|t-\tau|^{2H-2} = (t+z)^{2H-2} \tag{1}$$

where T is the time interval on which the traffic input data is given, z is the time interval on which the prediction is made, and H is the Hurst exponent. Eq. (1) is valid only if $0.5 < H < 1$, for simplicity we do not consider other situations. The integral equation (1) is the Wiener–Hopf integral equation. This equation may be solved via the Galerkin method, the idea of which is as follows [7].

The unknown weight function is sought in the form

$$h(\tau) = \sum_{j=1}^{n} g_j f_j(\tau) \tag{2}$$

where $f_j(\tau)$ are the functions orthogonal on the time interval $\tau \in [0, T]$ and g_j are the unknown coefficients. From (1) and (2) one can obtain

$$\sum_{j=1}^{n} g_j \int_0^T d\tau f_j(\tau)|t-\tau|^{2H-2} = (t+z)^{2H-2}. \tag{3}$$

On multiplying both sides of (3) by $f_k(t)$, $k = 1,2,..,n$ and integrating over t , one can obtain

$$\sum_{j=1}^{n} g_j \int_0^T \int_0^T dt d\tau f_j(\tau) f_k(t) |t-\tau|^{2H-2} = \int_0^T dt f_k(t)(t+z)^{2H-2}, \quad k = \overline{1,n}. \tag{4}$$

Let us introduce the following designations:

$$G_{jk} = \int_0^T \int_0^T dt d\tau f_k(t) f_j(\tau) |t-\tau|^{2H-2}, \quad B_k = \int_0^T dt f_k(t)(t+z)^{2H-2}, \tag{5}$$

the quantities G_{jk} are the integral brackets. So (4) can be rewritten as

$$\sum_{j=1}^{n} G_{jk} g_j = B_k, \quad k = \overline{1,n}. \tag{6}$$

The obtained expression (6) is a system of linear algebraic equations for the unknown coefficients g_j. It can be solved with the help of the matrix method. (6) can be rewritten in matrix form

$$Gg = B \tag{7}$$

where G is the matrix of the integral brackets, g is the column vector of the unknown coefficients, and B is the column vector of the free terms:

$$G = \begin{pmatrix} G_{11} & G_{12} & \dots & G_{1n} \\ G_{21} & G_{22} & \dots & G_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n,2} & \dots & G_{nn} \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}. \tag{8}$$

On the basis of (7) and (8) in matrix form we have

$$g = G^{-1} B. \tag{9}$$

The functions $f_j(\tau)$ in the expansion (2) form a complete orthogonal function system, which usually contains an infinite number of functions. Nevertheless, the number of functions in expansion (2) should be artificially truncated; otherwise, the system (6) would contain an infinite number of equations and could hardly be treated. The solution in the form (2) is called the solution in the n-function approximation.

In this paper we propose to choose the functions $f_j(\tau)$ as the Walsh functions. As is known [12], the Walsh functions form a complete orthogonal function system and may be defined with the help of the Hadamard matrices. The Hadamard matrices $H^{(2^m)}$ may be introduced in a recursive way:

$$H^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H^{(2^{m+1})} = \begin{pmatrix} H^{(2^m)} & H^{(2^m)} \\ H^{(2^m)} & -H^{(2^m)} \end{pmatrix}, \quad m \in \mathbb{N}. \tag{10}$$

The Walsh functions in the Hadamard numeration $\mathrm{walh}_k(t)$ are defined as follows on the time interval $t \in [0,T]$:

$$\mathrm{walh}_k(t) = \begin{cases} H_{k1}^{(2^m)}, t \in [0, T/2^m] \\ H_{k2}^{(2^m)}, t \in (T/2^m, 2T/2^m] \\ H_{k3}^{(2^m)}, t \in (2T/2^m, 3T/2^m] \\ \vdots \\ H_{k,2^m}^{(2^m)}, t \in ((2^m-1)T/2^m, T] \end{cases} \tag{11}$$

where $H_{jl}^{(2^m)}$ are the Hadamard matrix elements and m is the least natural number that obeys the inequality $k \leq 2^m$. The set of Walsh functions in the Walsh numeration coincides with that in the Hadamard numeration, but the numerations differ from each other. In the Walsh numeration, the Walsh functions $\mathrm{wal}_k(t)$ are numerated in ascending order of sign changes on the time interval $t \in (0,T)$. The first Walsh function $\mathrm{wal}_1(t) = 1 = \mathrm{const}$ has 0 sign changes, the second Walsh function

$$\mathrm{wal}_2(t) = \begin{cases} 1, t \in [0, T/2] \\ -1, t \in (T/2, T] \end{cases} \tag{12}$$

has 1 sign change, and so on. As is known [11], the so-called Walsh matrix $W^{(2^m)}$ may be formed in terms of the Hadamard matrix $H^{(2^m)}$ by rearranging the rows in ascending order of sign changes. For example, the matrices $H^{(4)}$ and $W^{(4)}$ are as follows:

$$H^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad W^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \tag{13}$$

so the first row in $W^{(4)}$ has 0 sigh changes, the second row in $W^{(4)}$ has 1 sign change, and so on. The Walsh functions $\mathrm{wal}_k(t)$ are defined as follows:

$$\mathrm{wal}_k(t) = \begin{cases} W_{k1}^{(2^m)}, t \in [0, T/2^m] \\ W_{k2}^{(2^m)}, t \in (T/2^m, 2T/2^m] \\ W_{k3}^{(2^m)}, t \in (2T/2^m, 3T/2^m] \\ \vdots \\ W_{k,2^m}^{(2^m)}, t \in ((2^m - 1)T/2^m, T] \end{cases} \tag{14}$$

where $W_{jl}^{(2^m)}$ are the Walsh matrix elements and m is the least natural number which obeys the inequality $k \leq 2^m$.

The Walsh matrix is built in some mathematical packages. In this paper we used the Wolfram Mathematica package, in the framework of which the Walsh matrix is expressed as

$$W^{(2^m)} = \sqrt{2^m}\mathrm{HadamardMartix}[2^m] \tag{15}$$

where the matrix HadamardMartix$[2^m]$ is built in the Wolfram Mathematica package.

In what follows, we use the Walsh functions in the Walsh numeration. We realize the Galerkin method with the help of a truncated Walsh function expansion. In other words, we put

$$f_j(t) = \mathrm{wal}_j(t)\ . \tag{16}$$

The use of the Walsh functions is convenient because they are step ones, which significantly simplifies the calculation in comparison with the polynomial solution derivation [8]. The derivation of the unknown weight function with the help of Walsh functions is given in the following section.

3. Derivation of the weight function

First of all, let us derive the integral brackets G_{jk} (see (5)). Let us consider the approximation of $n = 2^m$ Walsh functions. The Walsh functions $\mathrm{wal}_j(t)$ are step ones, they are constant on each time interval $t \in (lT/n, (l+1)T/n)$, $l = \overline{0, n-1}$. That is why the integral brackets (5) may be calculated, for example, as follows:

$$G_{jk} = \sum_{l,s=1}^{n} \theta_{jl}\theta_{ks}V_{ls} \tag{17}$$

where θ_{jl} are the corresponding values of the Walsh functions on the corresponding intervals:

$$\theta_{jl} = \mathrm{wal}_j\left(\frac{1}{2}\cdot\left(\frac{(l-1)T}{n} + \frac{lT}{n}\right)\right) = \mathrm{wal}_j\left(\frac{(2l-1)T}{2n}\right). \tag{18}$$

and V_{ls} are the following integrals:

$$V_{ls} = \int_{\frac{(l-1)T}{n}}^{lT/n} \int_{\frac{(s-1)T}{n}}^{sT/n} dt d\tau |t - \tau|^{2H-2}. \tag{19}$$

As can be seen from (14),

$$\mathrm{wal}_j\left(\frac{(2l-1)T}{2n}\right) = W_{jl}^{(n)}, \tag{20}$$

so the integral brackets (14) may be rewritten as

$$G_{jk} = \sum_{l,s=1}^{n} W_{jl}^{(n)} W_{ks}^{(n)} V_{ls}. \tag{21}$$

The integrals V_{ls} can be calculated analytically. The quantities V_{ls} have the following properties:

$$V_{ls} = V_{sl} \tag{22}$$

and

$$V_{ls} = V_{l+1,s+1}. \tag{23}$$

As for the property (22), let us redefine the variables:

$$\begin{gathered}
V_{ls} = \int\limits_{\frac{(l-1)T}{n}}^{lT/n} dt \int\limits_{\frac{(s-1)T}{n}}^{sT/n} d\tau |t-\tau|^{2H-2} = \{t \leftrightarrow \tau\} = \int\limits_{\frac{(l-1)T}{n}}^{lT/n} d\tau \int\limits_{\frac{(s-1)T}{n}}^{sT/n} dt |\tau - t|^{2H-2} = \\
= \{|\tau - t|^{2H-2} = |t-\tau|^{2H-2}\} = \int\limits_{\frac{(s-1)T}{n}}^{sT/n} dt \int\limits_{\frac{(l-1)T}{n}}^{lT/n} d\tau |t-\tau|^{2H-2} = V_{sl}.
\end{gathered} \tag{24}$$

As for the property (23),

$$\begin{gathered}
V_{ls} = \int\limits_{\frac{(l-1)T}{n}}^{lT/n} dt \int\limits_{\frac{(s-1)T}{n}}^{sT/n} d\tau |t-\tau|^{2H-2} = \left\{x = \frac{T}{n} + t, y = \frac{T}{n} + \tau\right\} = \\
= \int\limits_{lT/n}^{\frac{(l+1)T}{n}} dx \int\limits_{sT/n}^{\frac{(s+1)T}{n}} dy \left|\left(x - \frac{T}{n}\right) - \left(y - \frac{T}{n}\right)\right|^{2H-2} = \int\limits_{lT/n}^{\frac{(l+1)T}{n}} dx \int\limits_{sT/n}^{\frac{(s+1)T}{n}} dy |x-y|^{2H-2} = \\
= V_{l+1.s+1}.
\end{gathered} \tag{25}$$

On the basis of (22) and (23) one can conclude that the matrix of the quantities V_{ls} takes the form

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} & \cdots & V_{1n} \\ V_{12} & V_{11} & V_{12} & \cdots & V_{1,n-1} \\ V_{13} & V_{12} & V_{11} & \cdots & V_{1,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_{1n} & V_{1,n-1} & V_{1,n-2} & \cdots & V_{11} \end{pmatrix}, \tag{26}$$

so a straightforward calculation is needed only for the first row of the matrix V (n straightforward calculations rather than n^2) . Let us calculate V_{11}:

$$V_{11} = \int\limits_0^{T/n} dt \int\limits_0^{T/n} d\tau |t-\tau|^{2H-2} = \int\limits_0^{T/n} dt Y(t) \tag{27}$$

where

$$Y(t) = \int\limits_0^{\frac{T}{n}} d\tau |t-\tau|^{2H-2} = \{y = t - \tau\} = -\int\limits_t^{t-\frac{T}{n}} dy |y|^{2H-2} = \int\limits_{t-\frac{T}{n}}^{t} dy |y|^{2H-2} \tag{28}$$

As can be seen from (27), $t \in \left(0, \frac{T}{n}\right)$, so the limits of integration in the last expression in (28) have opposite signs, and

$$\begin{gathered}
Y(t) = \int\limits_{t-\frac{T}{n}}^{0} dy (-y)^{2H-2} + \int\limits_0^t dy y^{2H-2} = \{u = -y\} = \int\limits_0^{\frac{T}{n}-t} du u^{2H-2} + \int\limits_0^t dy y^{2H-2} = \\
= \frac{1}{2H-1}\left(\left(\frac{T}{n} - t\right)^{2H-1} + t^{2H-1}\right).
\end{gathered} \tag{29}$$

So, on the basis of (29) and (27) one can obtain

$$V_{11} = \frac{1}{2H-1}\int\limits_0^{T/n} dt \left(\frac{T}{n} - t\right)^{2H-1} + \frac{1}{2H-1}\int\limits_0^{T/n} dt t^{2H-1} = \left\{u = \frac{T}{n} - t\right\} = \tag{30}$$

$$= \frac{1}{2H-1}\int_{0}^{T/n} duu^{2H-1} + \frac{1}{2H-1}\int_{0}^{T/n} dtt^{2H-1} = \frac{1}{H(2H-1)}\left(\frac{T}{n}\right)^{2H}.$$

Let us calculate V_{1l}, $l = \overline{2,n}$:

$$V_{1l} = \int_{0}^{\frac{T}{n}} dt \int_{\frac{(l-1)T}{n}}^{\frac{lT}{n}} d\tau |t-\tau|^{2H-2} = \int_{0}^{\frac{T}{n}} dt X_l(t) \tag{31}$$

where

$$X_l(t) = \int_{\frac{(l-1)T}{n}}^{\frac{lT}{n}} d\tau |t-\tau|^{2H-2} = \{y = t-\tau\} = \int_{t-\frac{lT}{n}}^{t-\frac{(l-1)T}{n}} dy|y|^{2H-2} \tag{32}$$

As can be seen from (21), $t \in \left(0, \frac{T}{n}\right)$, so both limits of integration in the last expression in (32) are non-positive, and

$$\begin{aligned} X_l(t) &= \int_{t-\frac{lT}{n}}^{t-\frac{(l-1)T}{n}} dy(-y)^{2H-2} = \{u=-y\} = \int_{\frac{(l-1)T}{n}-t}^{\frac{lT}{n}-t} duu^{2H-2} = \\ &= \frac{1}{2H-1}\left(\left(\frac{lT}{n}-t\right)^{2H-1} - \left(\frac{(l-1)T}{n}-t\right)^{2H-1}\right). \end{aligned} \tag{33}$$

So on the basis of (33) and (31) one can conclude that

$$\begin{aligned} V_{1l} &= \frac{1}{2H-1}\int_{0}^{T/n} dt\left(\frac{lT}{n}-t\right)^{2H-1} - \frac{1}{2H-1}\int_{0}^{T/n} dt\left(\frac{(l-1)T}{n}-t\right)^{2H-1} = \\ &= \left\{y = \frac{lT}{n}-t, u = \frac{(l-1)T}{n}-t\right\} = \frac{1}{2H-1}\int_{\frac{(l-1)T}{n}}^{\frac{lT}{n}} dyy^{2H-1} - \frac{1}{2H-1}\int_{\frac{(l-2)T}{n}}^{\frac{(l-1)T}{n}} duu^{2H-1} = \\ &= \frac{1}{2H(2H-1)}\left(\left(\frac{lT}{n}\right)^{2H} + \left(\frac{(l-2)T}{n}\right)^{2H} - 2\left(\frac{(l-1)T}{n}\right)^{2H}\right). \end{aligned} \tag{34}$$

For clarity, let us summarize the results:

$$\begin{aligned} V_{11} &= \frac{1}{H(2H-1)}\left(\frac{T}{n}\right)^{2H}; \quad l = \overline{2,n}, \\ V_{1l} &= \frac{1}{2H(2H-1)}\left(\left(\frac{lT}{n}\right)^{2H} + \left(\frac{(l-2)T}{n}\right)^{2H} - 2\left(\frac{(l-1)T}{n}\right)^{2H}\right). \end{aligned} \tag{35}$$

Now we should calculate the free terms B_k (see (5)). On the basis of the same idea, one can rewrite

$$B_k = \sum_{s=1}^{n} W_{ks}^{(n)} Q_s \tag{36}$$

where

$$Q_s = \int_{\frac{(s-1)T}{n}}^{\frac{sT}{n}} dt(t+z)^{2H-2}. \tag{37}$$

The integral (37) can be analytically calculated:

$$Q_s = \{y = t + z\} = \int_{\frac{(s-1)T}{n}+z}^{\frac{sT}{n}+z} dy y^{2H-2} = \frac{1}{2H-1}\left[\left(\frac{sT}{n}+z\right)^{2H-1} - \left(\frac{(s-1)T}{n}+z\right)^{2H-1}\right]. \quad (38)$$

So

$$B_k = \frac{1}{2H-1}\sum_{s=1}^{n} W_{ks}^{(n)}\left[\left(\frac{sT}{n}+z\right)^{2H-1} - \left(\frac{(s-1)T}{n}+z\right)^{2H-1}\right]. \quad (39)$$

Finally, let us show that the integral brackets G_{jk} have the properties

$$G_{jk} = G_{kj} \quad (40)$$

and

$$G_{jk} = 0 \text{ if } j \text{ and } k \text{ have opposite parities.} \quad (41)$$

As for the property (40), on the basis of (5) and (16) we have

$$G_{jk} = \int_0^T\int_0^T dt d\tau \mathrm{wal}_k(t)\mathrm{wal}_j(\tau)|t-\tau|^{2H-2} = \{t \leftrightarrow \tau\} =$$
$$= \int_0^T\int_0^T d\tau dt \mathrm{wal}_k(\tau)\mathrm{wal}_j(t)|\tau - t|^{2H-2} = \int_0^T\int_0^T dt d\tau \mathrm{wal}_j(t)\mathrm{wal}_k(\tau)|t-\tau|^{2H-2} = G_{kj}. \quad (42)$$

To prove the property (41), first of all we should stress that the Walsh functions obey the following properties:

$$\mathrm{wal}_k\left(\frac{T}{2}-x\right) = \begin{cases} \mathrm{wal}_k\left(\frac{T}{2}+x\right), k \text{ is even} \\ -\mathrm{wal}_k\left(\frac{T}{2}+x\right), k \text{ is odd} \end{cases}. \quad (43)$$

So let us consider the quantities G_{jk} where j and k have opposite parities:

$$G_{jk} = \int_0^T\int_0^T dt d\tau \mathrm{wal}_k(t)\mathrm{wal}_j(\tau)|t-\tau|^{2H-2} = \left\{x = \frac{T}{2} - t, y = \frac{T}{2} - \tau\right\} =$$
$$= \int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt d\tau \mathrm{wal}_k\left(\frac{T}{2}-x\right)\mathrm{wal}_j\left(\frac{T}{2}-y\right)|y-x|^{2H-2} = \{\beta = -x, \gamma = -y\} =$$
$$= \int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt d\tau \mathrm{wal}_k\left(\frac{T}{2}+\beta\right)\mathrm{wal}_j\left(\frac{T}{2}+\gamma\right)|\beta-\gamma|^{2H-2} = \quad (44)$$
$$= -\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt d\tau \mathrm{wal}_k\left(\frac{T}{2}-\beta\right)\mathrm{wal}_j\left(\frac{T}{2}-\gamma\right)|\beta-\gamma|^{2H-2} = -G_{jk}$$

which leads to the property (41). It should be stressed the following fact is used in (44):

$$\mathrm{wal}_k\left(\frac{T}{2}-x\right)\mathrm{wal}_j\left(\frac{T}{2}-x\right) = -\mathrm{wal}_k\left(\frac{T}{2}+x\right)\mathrm{wal}_j\left(\frac{T}{2}+x\right), \quad (45)$$
$$j \text{ and } k \text{ have opposite parities,}$$

(45) follows from (43). So, a straightforward calculation is needed only for the integral brackets G_{jk} where $j \geq k$ and j, k are of the same parity.

To summarize the above-mentioned, let us write the algorithm of the weight function derivation in the approximation of $n = 2^m$ Walsh functions:

1. Calculate the quantities V_{11} and V_{1l}, $l = \overline{2,n}$ by formulas (35).
2. Form the matrix V from the elements V_{ls}, $l, s = \overline{1,n}$ by formula (26).

3. Make a straightforward calculation of the integral brackets G_{jk} by formula (21) for $j \geq k$ and j, k of the same parity. Calculate the other integral brackets on the basis of the properties (40), (41) and form the corresponding matrix G (see (8)).
4. Calculate the free terms B_k by formula (39) and form the column vector of the free terms B (see (8)).
5. Calculate the column vector of the coefficients g by formula (9)
6. Obtain the weight function $h(\tau)$ by formulas (2) and (16).

In contrast to the previous investigations [8], the proposed algorithm does not require the calculation of the integrals with the help of the mathematical packages – all the integrals are calculated analytically.

The following section contains a numerical comparison for both sides of the Wiener–Hopf integral equation for the obtained results.

4. Numerical results

In order to verify the above-mentioned algorithm, in this section we calculate the MAPE (mean average percentage error) of the residual (the difference of the left-hand and the right-hand sides) of the Wiener-Hopf integral equation for the obtained weight functions.

The left-hand side of the integral equation (1) is as follows:

$$\mathrm{Left}(t) = \int_0^T d\tau h(\tau)|t-\tau|^{2H-2} = \sum_{l=1}^{n} h_l X_l(t) \tag{46}$$

where

$$h_l = h\left(\frac{1}{2}\left(\frac{(l-1)T}{n} + \frac{lT}{n}\right)\right) = h\left(\frac{(2l-1)T}{2n}\right) \tag{47}$$

and

$$X_l(t) = \int_{\frac{(l-1)T}{n}}^{\frac{lT}{n}} d\tau |t-\tau|^{2H-2} = \int_{t-\frac{lT}{n}}^{t-\frac{(l-1)T}{n}} dy|y|^{2H-2}, \tag{48}$$

see (32), here the fact that $h(\tau)$ is a step function is used.

Let us consider different cases:

1. $\frac{(l-1)T}{n} < t < \frac{lT}{n}$. In such a case, the limits of integration in (48) are of opposite signs, and

$$X_l(t) = \int_{t-\frac{lT}{n}}^{0} dy(-y)^{2H-2} + \int_0^{t-\frac{(s-1)T}{n}} dy y^{2H-2} = \{u=-y\} =$$
$$= \int_0^{\frac{lT}{n}-t} du u^{2H-2} + \int_0^{t-\frac{(s-1)T}{n}} dy y^{2H-2} = \frac{1}{2H-1}\left(\left(\frac{lT}{n}-t\right)^{2H-1} + \left(t-\frac{(l-1)T}{n}\right)^{2H-1}\right). \tag{49}$$

2. $t \geq \frac{lT}{n}$. In such a case, both limits of integration in (48) are non-negative, and

$$X_l(t) = \int_{t-\frac{lT}{n}}^{t-\frac{(l-1)T}{n}} dy y^{2H-2} = \frac{1}{2H-1}\left(\left(t-\frac{(l-1)T}{n}\right)^{2H-1} - \left(t-\frac{lT}{n}\right)^{2H-1}\right) \tag{50}$$

3. $t \leq \frac{(l-1)T}{n}$. In such a case, both limits of integration in (48) are non-positive, and $X_l(t)$ is given by formula (33).

To summarize the above-mentioned,

$$X_l(t)=\begin{cases}\frac{1}{2H-1}\left(\left(\frac{lT}{n}-t\right)^{2H-1}+\left(t-\frac{(l-1)T}{n}\right)^{2H-1}\right), \frac{(l-1)T}{n}<t<\frac{lT}{n}\\ \frac{1}{2H-1}\left(\left(t-\frac{(l-1)T}{n}\right)^{2H-1}-\left(t-\frac{lT}{n}\right)^{2H-1}\right), t\geq\frac{lT}{n}\\ \frac{1}{2H-1}\left(\left(\frac{lT}{n}-t\right)^{2H-1}-\left(\frac{(l-1)T}{n}-t\right)^{2H-1}\right), t\leq\frac{(l-1)T}{n}\end{cases}. \quad (51)$$

The right-hand side of the Wiener–Hopf integral equation (1) is as follows:

$$\mathrm{Right}(t)=(t+z)^{2H-2}. \quad (52)$$

The MAPE may be introduced as follows:

$$\mathrm{MAPE}=\frac{1}{T}\int_0^T\left|\frac{\mathrm{Left}(t)-\mathrm{Right}(t)}{\mathrm{Right}(t)}\right|dt\cdot 100\%, \quad (53)$$

The integral (53) is the only integral that cannot be calculated analytically. We use the method of trapezoids in order to calculate an approximate value of (53):

$$\begin{aligned}\mathrm{MAPE}&\approx\frac{1}{T}\cdot\sum_{j=1}^{N}\left|\frac{\mathrm{Left}((2j-1)T/2N)-\mathrm{Right}((2j-1)T/2N)}{\mathrm{Right}((2j-1)T/2N)}\right|\cdot\frac{T}{N}\cdot 100\%=\\&=\sum_{j=1}^{N}\left|\frac{\mathrm{Left}((2j-1)T/2N)-\mathrm{Right}((2j-1)T/2N)}{\mathrm{Right}((2j-1)T/2N)}\right|\cdot\frac{100\%}{N}\end{aligned} \quad (54)$$

where N is the number of intervals into which the interval $(0,T)$ is divided. In this paper, $N=10^4$.

To summarize the above-mentioned, let us write the algorithm of calculation of the MAPE for the derived weight function in the approximation of $n=2^m$ Walsh functions:

1. Calculate the quantities h_l by formula (47)
2. Introduce the function $\mathrm{Left}(t)$ by formulas (51) and (46)
3. Introduce the function $\mathrm{Right}(t)$ by formula (52)
4. Calculate the MAPE by formula (54)

In what follows, some numerical results are given. The following parameters were considered:

$$T=100, \qquad z=3, \qquad H=0.8, \quad (55)$$

it should be stressed that these parameters were investigated in our previous work [8] devoted to polynomial solutions. The MAPEs for parameters (55) are given in Table 1.

Table 1

MAPE for parameters (55) for approximations of different numbers of Walsh functions

Number of Walsh functions	MAPE,%
2	15
4	6.4
8	2.6
16	1.0
32	$3.9\cdot10^{-1}$
64	$1.5\cdot10^{-1}$
128	$6.0\cdot10^{-2}$
256	$2.4\cdot10^{-2}$

The MAPE values in Table 1 are rounded off to two significant digits. The corresponding graphs are given for the approximation of 256 Walsh functions (see Fig. 1).

The following parameters were also considered:

$$T=1000, \qquad z=3, \qquad H=0.8, \quad (56)$$

The MAPEs for parameters (56) are given in Table 2.

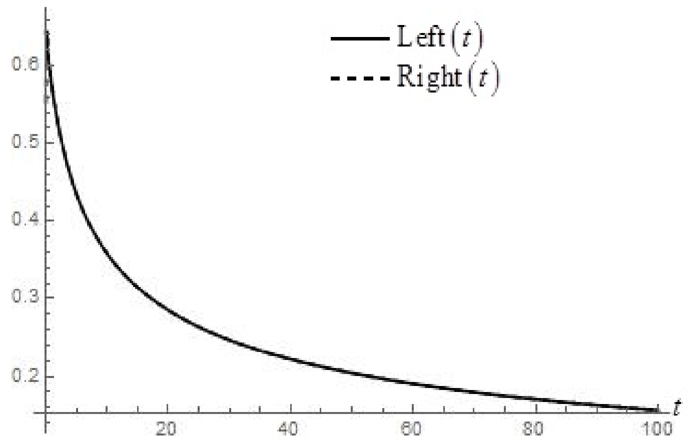

Figure 1: Both sides of the integral equation (1) for the approximation of 256 Walsh functions for parameters (55)

Table 2
MAPE for parameters (56) for approximations of different numbers of Walsh functions

Number of Walsh functions	MAPE,%
2	19
4	9.4
8	4.4
16	2.0
32	$8.5 \cdot 10^{-1}$
64	$3.5 \cdot 10^{-1}$
128	$1.3 \cdot 10^{-1}$
256	$5.1 \cdot 10^{-2}$

The MAPE values in Table 2 are also rounded off to two significant digits, the graphs for the approximation of 256 Walsh functions is given on Fig. 2.

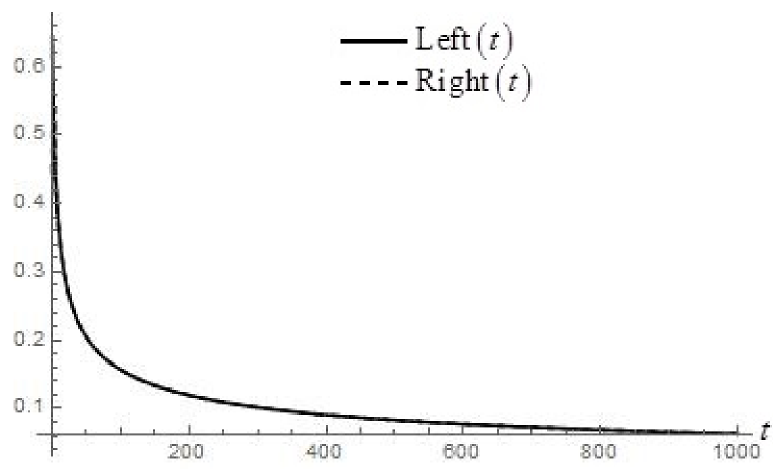

Figure 2: Both sides of the integral equation (1) for the approximation of 256 Walsh functions for parameters (56)

As can be seen from Table 1 and Table 2, the approximations of rather small numbers of Walsh functions are not accurate, but the approximations of rather large numbers of Walsh functions are rather accurate. Both sides of the integral equation (1) almost coincide for the approximation of 256 Walsh functions.

The investigation of the polynomial solutions [8] for the parameters (55) is limited by the approximation of 19 polynomials only (the corresponding MAPE is equal to 0.57%), the Wolfram

Mathematica is not able to build a graph of the left-hand side of the integral equation adequately for the approximations of a number of polynomials greater than 19. In our opinion, this is because of the products of very large and very small numbers. The method based on the Walsh functions does not have such a disadvantage, and the approximations of a few hundreds of Walsh functions may be investigated, the corresponding MAPE values are less than 0.57%. The parameters (56) were not investigated in [8], their investigation is given in order to illustrate that the proposed method based on the Walsh functions may be applied in a rather wide range of parameters.

5. Conclusions

The problem of prediction of telecommunication traffic is important for telecommunications; in particular, it is important for information security because security attacks may be detected if the traffic behavior significantly differs from the predicted one [1].

We investigate the theoretical fundamentals of the Kolmogorov–Wiener filter construction for the continuous telecommunication traffic prediction. Our goal is to develop a method of the filter weight function derivation as a solution of the Wiener–Hopf integral equation. The traffic is taken in the model where it is treated as continuous fractional Gaussian noise.

The corresponding Wiener–Hopf integral equation for the unknown weight function is solved via the Galerkin method. The idea of the Galerkin method is to seek the unknown weight function in the form of an expansion into an artificially truncated series in orthogonal functions. In our previous work [8] we used the Galerkin method based on the polynomial functions. In this paper we use the Walsh functions instead of the polynomial ones, which leads to the following advantages:

1. The integral brackets can be obtained analytically, and the corresponding analytical expressions are applicable even for a rather large number of Walsh functions; the numerical calculation of the double integrals in the integral brackets is not needed.
2. An easy-to-use analytical expression for the function $\mathrm{Left}(t)$, which is the left-hand side of the corresponding Wiener-Hopf integral equation, may be obtained; the numerical calculation of the corresponding integral is not needed.
3. The products of very large and very small numbers are absent in the framework of the proposed method based on the Walsh functions, which allows one to investigate the approximations of large numbers of Walsh functions.

It should also be noted that the only integral that should be calculated numerically is the integral (53) for the MAPE. The other integrals are calculated analytically.

The approximations of $n = 2^m$ Walsh functions are investigated both for the parameters (55) and (56). The investigation is conducted up to the approximation of 256 Walsh functions. For comparison, it should be noted that the investigation of polynomial solutions [8] for the parameters (55) is limited by the approximation of 19 polynomials only, the Wolfram Mathematica is not able to treat the approximations of more polynomials adequately. The parameters (56) were not investigated on the basis of polynomial functions, the corresponding investigation on the basis of Walsh functions is given in order to stress that the proposed Walsh function approach is applicable in a rather wide range of parameters.

The accuracy of the approximations rises with the number of Walsh functions, and the coincidence of both sides of the integral equation under consideration is very good for rather large numbers of Walsh functions.

It should be noted that in this paper we investigate only the theoretical fundamentals of the Kolmogorov–Wiener filter construction. This filter may be applied to the prediction of stationary processes, so it is logical enough to use the approach based on this filter for the prediction of the stationary telecommunication traffic, for example, in the model where the traffic is treated as fractional Gaussian noise and in the model where the traffic is treated as a process with a power-law structure function. There are plenty of different approaches to traffic prediction [1], for example, the ARIMA models, the approaches based on the wavelet transforms, the approaches based on the neural networks and so on, but the approach based on the Kolmogorov–Wiener filter is not sufficiently developed in the literature. The Kolmogorov–Wiener filter is linear and stationary, so it is a rather simple filter, and the use of the Kolmogorov–Wiener filter may be less complicated than the use of the above-mentioned approaches. That is why the proposed approach may have practical significance for the prediction of stationary telecommunication traffic. The concrete experimental prediction of the modeled or real telecommunication traffic based on the proposed approach is our plan for the future and may be given in another paper.

One more plan for the future is the application of the developed approach to other stationary traffic models. For example, the polynomial solutions need a significant enhancement for the power-law structure function model (see [11]), so the application of the developed approach to the traffic prediction in the framework of that model may be another plan for the future.

6. References

[1] M. F. Iqbal, M. Zahid, D. Habib, L. K. John, Efficient Prediction of Network Traffic for Real-Time Applications, Journal of Computer Networks and Communications 2019 (2019) 4067135. doi:10.1155/2019/4067135.

[2] S. Lysenko, K. Bobrovnikova, S. Matiukh, I. Hurman & O. Savenko, Detection of the botnets' low-rate DDoS attacks based on self-similarity. International Journal of Electrical & Computer Engineering, 2020, 10, 2088-8708. DOI: http://doi.org/10.11591/ijece.v10i4.pp3651-3659.

[3] D. Anderson, W. S. Cleveland, B. Xi, Multifractal and Gaussian fractional sum–difference models for Internet traffic, Performance Evaluation 107 (2017) 1–33. doi:10.1016/j.peva.2016.11.001.

[4] K. Filus, A. Domanski, J. Domanska, D. Marek, J. Szygula, Long-Range Dependent Traffic Classification with Convolutional Neural Networks Based on Hurst Exponent Analysis, Entropy 22 (2020) 1159. doi:10.3390/e22101159.

[5] V. Kh. Bagmanov, A. M. Komissarov, A. Kh. Sultanov, Teletraffic forecast on the basis of fractal fliters, Bulletin of Ufa State Aviation Technical University 9 (2007) 217–222. (in Russian).

[6] S. L. Miller, D. Childers, Probability and Random Processes With Applications to Signal Processing and Communications, 2nd. ed., Elsevier, Academic Press, Amsterdam, 2012, doi: 10.1016/C2010-0-67611-5.

[7] A. D. Polyanin, A. V. Manzhirov, Handbook of integral equations, 2nd. ed., Chapman & Hall/CRC Press, Taylor & Francis Group, Boca Raton, 2008, doi: 10.1201/9781420010558.

[8] V. Gorev, A. Gusev, V, Korniienko, Fractional Gaussian noise traffic forecasting based on the Walsh functions, CEUR Workshop Proceedings 2623 (2020) 164–173.

[9] A. I. Sokolovsky, S. A. Sokolovsky, O. A. Hrinishyn, On relaxation processes in a completely ionized plasma, East European Journal of Physics 3 (2020) 19-30, doi: 10.26565/2312-4334-2020-3-03

[10] R. G. Gonzalez, N. Khalil, V. Garzo, Enskog kinetic theory for multicomponent granular suspensions, Physical Review E 101 (2020) 012904, doi: 10.1103/PhysRevE.101.012904

[11] V. Gorev, A. Gusev, V, Korniienko, Polynomial solutions for the Kolmogorov–Wiener filter weight function for fractal processes, Radio Electronics, Computer Science, Control 2 (2019) 44–52, doi: 10.15588/1607-3274-2019-2-5.

[12] G. Lindfield, J. Penny, Numerical Methods Using MATLAB, 4th. ed., Elsevier, Academic Press, Amsterdam, 2019, doi: 10.1016/C2016-0-00395-9