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On Properties of Epistemic State Mappings among Ranking Functions and Total Preorders

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Abstract. Ranking functions and total preorders on worlds are two common models for epistemic states that can represent conditional beliefs. To further explore the connection between these frameworks, we consider mappings among models of both frameworks. Especially interesting are mappings that preserve desirable properties like syntax splittings, or are compatible with operations like marginalization and conditionalization. In this paper, we introduce postulates for such mappings. We evaluate the postulates for mappings within and across the two frameworks, establishing dependencies as well as incompatibilities among the postulates. The results will be useful for transferring methods developed for ranking functions to the total-preorder framework and the other way round.

1 Introduction

In the field of knowledge representation, there is a long tradition of employing conditionals as fundamental objects. A conditional formalizes a defeasible rule "If A then usually B" for logical formulas A, B and is often denoted as $(B|A)$. As conditional logic is more expressive than propositional logic, it requires a richer semantics as well. There are different approaches to semantics for conditional logic, e.g., [18, 1, 17, 20, 6, 4, 14]. These approaches often use either some form of ranking functions [23] or total preorders on interpretations as models for conditionals and conditional belief bases.

In this paper, we focus on these two kinds of models for conditionals, ranking functions (or ordinal conditional functions, OCFs) and total preorders on worlds (TPOs). Both models have their own advantages. TPOs are used in characterisation theorems for AGM revisions [13] as well as system P inference [1, 17]. OCFs allow to model the strength of conditional beliefs by assigning numbers to logical interpretations [23, 9]. Furthermore, some belief revision operators with interesting properties have been defined for OCFs, e.g., [14]. To better understand the connection between OCFs and TPOs, we investigate transformations among these frameworks, i.e., functions that map OCFs to TPOs or TPOs to

OCFs. Furthermore, we generalize by also including transformations from OCFs to OCFs and TPOs to TPOs.

We formalize functions on these models within and across the two frameworks as *epistemic state mappings* and propose postulates that govern epistemic state mappings. The postulates require the epistemic state mappings to preserve certain properties of the models like the entailed inference relation and syntax splittings. Syntax splitting is a concept describing that beliefs about different parts of the signature are uncorrelated [19, 21]. Other postulates ensure compatibility with the operations marginalization and conditionalization. These operations are relevant e.g. for some forms of forgetting [5, 7, 3], syntax splitting, and some aspects of belief revision [15, 22]. We investigate relationships among our postulates in general as well as for each framework in particular. Our results elaborate dependencies among the postulates, and they also unveil situations where certain combinations of postulates cannot be satisfied simultaneously.

In summary, the main contributions of this paper are:

- Introduction of epistemic state mappings for TPOs and OCFs
- Coverage of marginalization and conditionalization for the iterated case via the introduction of restricted TPOs and restricted OCFs
- Formalization of desirable properties of epistemic state mappings in terms of general postulates
- Establishment of relationships among the postulates and of realizability results for the postulates and for subsets thereof.

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The following text is structured as follows. After giving some background on conditional logic, ranking functions and total preorders in Section 2, we introduce marginalization, conditionalization, and syntax splitting in Section 3. We proceed to introduce the concept of epistemic state mappings and postulates for such mappings in Section 4. Then, we analyse the relationship among the postulates for epistemic state mappings from TPOs to TPOs in Section 5.1 and among the postulates for epistemic state mappings from OCFs to OCFs in Section 5.2. In Section 5.3, we consider epistemic state mappings from OCFs to TPOs, and in Section 5.4 we consider epistemic state mappings from TPOs to OCFs. In Section 6, we conclude and point out future work.

2 Background: Conditional Logic, Ranking Functions, and Total Preorders

A *(propositional) signature* is a finite set Σ of identifiers. For a signature Σ, we denote the propositional language over Σ by $\mathcal{L}_\Sigma$. Usually, we denote elements of the signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. The set of interpretations over a signature Σ is denoted as Ω_Σ. Interpretations are also called *worlds* and Ω_Σ is called the *universe*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}_\Sigma$ if A holds in ω. This is denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $Mod_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$. A formula A *entails* a formula B if $Mod_\Sigma(A) \subseteq Mod_\Sigma(B)$. Two formulas A and B are *equivalent*, denoted as $A \equiv B$, if they have the same models, i.e., $Mod_\Sigma(A) = Mod_\Sigma(B)$.

A *conditional* $(B|A)$ connects two formulas A, B and represents the rule "If A then usually B". For a conditional $(B|A)$ the formula A is called the *antecedent* and the formula B the *consequent* of the conditional. The conditional language over a signature Σ is denoted as $(\mathcal{L}|\mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}_\Sigma\}$. The set $(\mathcal{L}|\mathcal{L})_\Sigma$ is a flat conditional language as it does not allow nesting conditionals.

We use a three-valued semantics of conditionals in this paper [8]. For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Conditionals are usually considered in the context of epistemic states. An *epistemic state* is a structure that represents all beliefs that are relevant for an agent's reasoning.

There exist different kinds of models for epistemic states that can handle conditionals. Two approaches to this are ranking functions and total preorders on possible worlds.

A *ranking function* [23], also called *ordinal conditional function* (OCF), is a function $\kappa : \Omega_\Sigma \to \mathbb{N}_0 \cup \{\infty\}$ such that $\kappa^{-1}(0) \neq \emptyset$. The intuition of an OCF is that the rank of a world is lower if the world is more plausible. Therefore, OCFs can be seen as some kind of "implausibility measure". OCFs are extended to formulas by $\kappa(A) = \min_{\omega \in Mod(A)} \kappa(\omega)$ with $\min_\emptyset(\ldots) = \infty$. An OCF κ models a conditional $(B|A)$, denoted as $\kappa \models (B|A)$ if $\kappa(AB) < \kappa(A\overline{B})$, i.e., if the verification of the conditional is strictly more plausible than its falsification. The uniform ranking function κ_{uni} with $\kappa_{\text{uni}}(\omega) = 0$ for every $\omega \in Mod_\Sigma(A)$ represents the state of complete ignorance.

A *total preorder* (TPO) is a total, reflexive, and transitive binary relation. The meaning of a total preorder $\preceq$ on Ω_Σ as model for an epistemic state is that ω_1 is at least as plausible as ω_2 if $\omega_1 \preceq \omega_2$ for $\omega_1, \omega_2 \in \Omega_\Sigma$. The strict version of a TPO $\preceq$ is the relation $\prec$ defined by $\omega_1 \prec \omega_2$ if $\omega_1 \preceq \omega_2$ and $\omega_2 \not\preceq \omega_1$. TPOs on worlds are extended to consistent formulas by $A \preceq B$ if $\min(Mod_\Sigma(A), \preceq) \preceq \min(Mod_\Sigma(B), \preceq)$. A TPO $\preceq$ models a conditional $(B|A)$, denoted as $\preceq \models (B|A)$, if $AB \prec A\overline{B}$, i.e., if the verification of the conditional is strictly more plausible than its falsification.

3 Marginalization, Conditionalization, Syntax Splitting

We want to consider transformations among models of epistemic states represented by ranking functions or total preorders. To establish a notion for the domain of such transformations, we define the sets $\mathcal{M}_{TPO}(\Sigma)$ and $\mathcal{M}_{OCF}(\Sigma)$ containing all models over a certain signature Σ:

$$\mathcal{M}_{TPO}(\Sigma) = \{\preceq \subseteq \Omega_\Sigma \times \Omega_\Sigma \mid \preceq \text{ total preorder over } \Omega_\Sigma\}$$
$$\mathcal{M}_{OCF}(\Sigma) = \{\kappa : \Omega_\Sigma \mapsto \mathbb{N}_0 \cup \{\infty\} \mid \kappa \text{ ranking function}\}$$

3.1 Conditionalization and Marginalization on TPOs and OCFs

Two operations on epistemic states that we will use in this paper are conditionalization and marginalization. Conditionalization restricts the set of worlds that are considered in an epistemic state. After the conditionalization with a formula A the resulting state only considers the elements of $Mod_\Sigma(A)$ as possible worlds.

A notion of conditionalization for TPOs where the models of $\overline{A}$ are shifted to the uppermost layer has been introduced in [15]. Here, we will use the concept of conditionalization where the models of $\overline{A}$ are removed entirely from the epistemic state. To capture the outcome of such a conditionalization, we extend the notion of OCFs and TPOs.

Definition 1 (restricted OCF/TPO). *A* restricted ranking function *over a set $M \subseteq \Omega_\Sigma$ is a function $\kappa : M \to \mathbb{N}_0 \cup \{\infty\}$ such that $\kappa^{-1}(0) \neq \emptyset$. Restricted ranking functions are extended to formulas by $\kappa(A) = \min_{\omega \in Mod(A) \cap M} \kappa(\omega)$ with $\min_\emptyset(\ldots) = \infty$.*

A TPO *$\preceq$ on a set $M \subseteq \Omega_\Sigma$ as model for an epistemic state is also called a* restricted total preorder*. Restricted total preorders on worlds are extended to formulas by $A \preceq B$ if $\min(Mod_\Sigma(A) \cap M, \preceq) \preceq \min(Mod_\Sigma(B) \cap M, \preceq)$.*

The intuition of restricted OCFs and TPOs is the same as for usual OCFs and TPOs: Worlds with lower rank or position in the ordering are more plausible. For a signature Σ, a formula $A \in \mathcal{L}_\Sigma$ and with $M_A = Mod_\Sigma(A)$ we define

$$\mathcal{M}_{TPO}(\Sigma, A) = \{\preceq \subseteq M_A \times M_A \mid \preceq \text{ total preorder over } M_A\}$$
$$\mathcal{M}_{OCF}(\Sigma, A) = \{\kappa : M_A \to \mathbb{N}_0 \cup \{\infty\} \mid \kappa \text{ ranking function}\}.$$

The restricted OCFs and TPOs properly include the original notation as $\mathcal{M}_I(\Sigma) = \mathcal{M}_I(\Sigma, \top)$ for $I \in \{TPO, OCF\}$. For state $\Psi \in \mathcal{M}_I(\Sigma, A)$, we call $sig(\Psi) = \Sigma$ the signature of Ψ and $dom(\Psi) = Mod_\Sigma(A)$ the domain of Ψ. Now we can define conditionalization using restricted OCFs/TPOs. This definition of conditionalization is based on the usual notion of conditionalization [23, 22] but was extended to cover restricted TPOs and OCFs as well.

Definition 2 (conditionalization of restricted OCFs/TPOs). *The* conditionalization *of ranking functions over $Mod_\Sigma(B)$ to the models of a formula $A \in \mathcal{L}_\Sigma$ is a function $\mathcal{M}_{OCF}(\Sigma, B) \to \mathcal{M}_{OCF}(\Sigma, A \wedge B)$, $\kappa \mapsto \kappa|A$ such that $\kappa|A(\omega) = \kappa(\omega) - \kappa(A)$ for $\omega \in Mod_\Sigma(A \wedge B)$.*

The conditionalization *of total preorders over $Mod_\Sigma(B)$ to the models of a formula $A \in \mathcal{L}_\Sigma$ is a function $\mathcal{M}_{TPO}(\Sigma, B) \to \mathcal{M}_{TPO}(\Sigma, A \wedge B)$, $\preceq \mapsto \preceq|A$ such that $\omega_1 \preceq|A\, \omega_2$ iff $\omega_1 \preceq \omega_2$ for $\omega_1, \omega_2 \in Mod_\Sigma(A \wedge B)$.*

Marginalization on the other hand restricts the epistemic state to a sub-signature of the original signature. We extended the notion of marginalization in [2, 16] to cover restricted TPOs and OCFs.

Definition 3 (marginalization). *The* marginalization *of OCFs from a signature Σ to a sub-signature $\Sigma' \subseteq \Sigma$ is a function $\mathcal{M}_{OCF}(\Sigma) \to \mathcal{M}_{OCF}(\Sigma')$, $\kappa \mapsto \kappa_{|\Sigma'}$ such that $\kappa_{|\Sigma'}(\omega) = \kappa(\omega)$ for $\omega \in \Omega_{\Sigma'}$ [23, 2].*

The marginalization *of TPOs from a signature* Σ *to a sub-signature* $\Sigma' \subseteq \Sigma$ *is a function* $\mathcal{M}_{TPO}(\Sigma) \to \mathcal{M}_{TPO}(\Sigma')$, $\preceq \mapsto \preceq_{|\Sigma'}$ *such that* $\omega_1 \preceq_{|\Sigma} \omega_2$ *iff* $\omega_1 \preceq \omega_2$ *for* $\omega_1, \omega_2 \in \Omega_{\Sigma'}$ *[2, 16].*

Note that a world ω of a sub-signature $\Sigma' \subseteq \Sigma$ is considered as a formula when evaluated in the context of Σ, i.e., the rank of $\omega \in \Omega_{\Sigma'}$ with respect to an OCF κ over Σ is the rank of the world $\omega' \in \Omega_{\Sigma}$ with the lowest rank that coincides with ω on the variables in Σ'.

The marginalizations of OCFs and TPOs presented above are special cases of general forgetful functors $Mod(\varrho)$ from Σ-models to Σ'-models given in [2] where $\Sigma' \subseteq \Sigma$ and ϱ is the inclusion from Σ' to Σ. Informally, a forgetful functor forgets everything about the interpretation of the symbols in $\Sigma \setminus \Sigma'$ when mapping a Σ-model to a Σ'-model.

For $\mathcal{M}_I(\Sigma, \top) = \mathcal{M}_I(\Sigma)$, the marginalization/conditionalization of the restricted OCFs/TPOs coincides with the marginalization/conditionalization of OCFs/TPOs.

3.2 Syntax Splitting

Syntax splitting was first introduced as property of belief sets in [19]. The basic idea is that a belief set contains independent information over different parts of the signature. The partition of the signature in these parts is called a syntax splitting for the considered belief set. Syntax splittings are useful properties of epistemic states, as they indicate that different parts of the state can be processed independently of each other.

The notion of syntax splitting was extended to other representations of epistemic states such as TPOs and OCFs in [16]. Postulates for processing TPOs and ranking functions have been introduced in [16, 12, 11]. We adapted these definitions to cover restricted TPOs/OCFs as well. For a partitioning $\{\Sigma_1, \ldots, \Sigma_n\}$ of a signature Σ and a world $\omega \in \Omega_\Sigma$, the world $\omega^j \in \Omega_{\Sigma_j}$ denotes the variable assignment of the variables in Σ_j as in ω in the following definitions.

Definition 4 (syntax splitting for restricted TPOs (adapted from [16])). *Let* $\preceq$ *be a TPO in* $\mathcal{M}_{TPO}(\Sigma, A)$. *Let* $\{\Sigma_1, \ldots, \Sigma_n\}$ *be a partitioning of* Σ *and* $\omega^{\neq i} := \bigwedge_{\substack{j=1,\ldots,n \\ i \neq j}} \omega^j$ *for* $\omega \in \Omega$ *and* $i = 1, \ldots, n$. *The partitioning* $\{\Sigma_1, \ldots, \Sigma_n\}$ *is a* syntax splitting *for* $\preceq$ *if*

- *there are formulas* $A_1, \ldots, A_n$ *such that* $A \equiv A_1 \wedge \cdots \wedge A_n$ *and* $A_i \in \Sigma_i$ *for* $i = 1, \ldots, n$
- *and, for* $i = 1, \ldots, n$ *and* $\omega_1, \omega_2 \in dom(\preceq)$,

$$\omega_1^{\neq i} = \omega_2^{\neq i} \quad \textit{implies} \quad \big(\omega_1 \preceq \omega_2 \textit{ iff } \omega_1^i \preceq_{|\Sigma_i} \omega_2^i\big).$$

A syntax splitting $\{\Sigma_1, \ldots \Sigma_n\}$ *is denoted by* $\Sigma = \Sigma_1 \dot{\cup} \cdots \dot{\cup} \Sigma_n$.

Definition 5 (syntax splitting for restricted OCFs (adapted from [16])). *Let* κ *be an OCF in* $\mathcal{M}_{OCF}(\Sigma, A)$. *A partitioning* $\{\Sigma_1, \ldots, \Sigma_n\}$ *of* Σ *is a* syntax splitting *for* κ *if*

– *there are formulas* $A_1, \dots, A_n$ *such that* $A \equiv A_1 \wedge \dots \wedge A_n$ *and* $A_i \in \Sigma_i$ *for* $i = 1, \dots, n$
– *and there are ranking functions* $\kappa_i \in \mathcal{M}_{OCF}(\Sigma_i, A_i)$ *for* $i = 1, \dots, n$ *such that* $\kappa(\omega) = \kappa_1(\omega^1) + \dots + \kappa_n(\omega^n)$ *for* $\omega \in dom(\kappa)$.

This is denoted as $\kappa = \kappa_1 \oplus \dots \oplus \kappa_n$.

The definitions of syntax splitting for restricted TPOs/OCFs yield the original definition of syntax splitting for TPOs/OCFs by considering $\mathcal{M}_I(\Sigma, \top)$.

4 Postulates for Mappings on Epistemic States

To formalize the transformations among models of epistemic states we introduce epistemic state mappings.

Definition 6. *Let* $I_1, I_2 \in \{TPO, OCF\}$. *An* epistemic state mapping *from* I_1 *to* I_2, *denoted as* $\xi : I_1 \rightsquigarrow I_2$, *is a function family* $\xi = (\xi_{\Sigma,A})$ *for signatures* Σ *and formulas* $A \in \mathcal{L}_\Sigma$ *with* $\xi_{\Sigma,A} : \mathcal{M}_{I_1}(\Sigma, A) \to \mathcal{M}_{I_2}(\Sigma, A)$ *such that* $A \equiv B$ *implies* $\xi_{\Sigma,A} = \xi_{\Sigma,B}$.

Example 1. The family of functions $\xi^{reverse}$ that reverses every TPO defined by $\xi^{reverse}_{\Sigma,A}(\preceq) = \preceq'$ with $\omega_1 \preceq' \omega_2$ iff $\omega_2 \preceq \omega_1$ for a signature Σ, $A \in \mathcal{L}_\Sigma$, $\preceq \in \mathcal{M}_{TPO}(\Sigma, A)$, and $\omega_1, \omega_2 \in Mod_\Sigma(A)$ is an epistemic state mapping from TPOs to TPOs.

Every epistemic state mapping represents a way to transform epistemic states of kind I_1 to epistemic states of kind I_2. Desirable properties of epistemic state mappings $(\xi_{\Sigma,A})$ can be stated in the form of postulates. Some of these postulates use the fact that both OCFs and TPOs induce a TPO on their domain. For a TPO $\preceq$, the induced ordering $\leq_{\preceq}$ is the order $\preceq$ itself. For an OCF κ, the induced ordering $\leq_\kappa$ is given by $\omega_1 \leq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$ for $\omega_1, \omega_2 \in dom(\kappa)$.

Postulates. *Let* $I_1, I_2 \in \{TPO, OCF\}$ *and let* $(\xi_{\Sigma,A})$ *be an epistemic state mapping from* I_1 *to* I_2. *Let* Σ *be a signature and* $A \in \mathcal{L}_\Sigma$, *and* $\Psi \in \mathcal{M}_{I_1}(\Sigma, A)$.

Let $(C|D) \in (\mathcal{L} \mid \mathcal{L})_\Sigma$.

(IE) $\Psi \models (C|D)$ *iff* $\xi_{\Sigma,A}(\Psi) \models (C|D)$.
(wIE$^\Rightarrow$) $\Psi \models (C|D)$ *implies* $\xi_{\Sigma,A}(\Psi) \models (C|D)$.
(wIE$^\Leftarrow$) $\xi_{\Sigma,A}(\Psi) \models (C|D)$ *implies* $\Psi \models (C|D)$.

Let ω_1, ω_2 *in* $dom(\Psi)$.

(Ord) $\omega_1 <_\Psi \omega_2$ *iff* $\omega_1 <_{\xi_{\Sigma,A}(\Psi)} \omega_2$.
(wOrd$^\Rightarrow$) $\omega_1 <_\Psi \omega_2$ *implies* $\omega_1 <_{\xi_{\Sigma,A}(\Psi)} \omega_2$.
(wOrd$^\Leftarrow$) $\omega_1 <_{\xi_{\Sigma,A}(\Psi)} \omega_2$ *implies* $\omega_1 <_\Psi \omega_2$.

(SynSplit) *If* $sig(\Psi) = \Sigma_1 \dot\cup \cdots \dot\cup \Sigma_n$ *is a syntax splitting for* Ψ*, then* $\Sigma_1 \dot\cup \cdots \dot\cup \Sigma_n$ *is a syntax splitting for* $\xi_{\Sigma,A}(\Psi)$.

(SynSplit$^\mathrm{b}$) *If* $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ *is a syntax splitting for* Ψ*, then* $\Sigma_1 \dot\cup \Sigma_2$ *is a syntax splitting for* $\xi_{\Sigma,A}(\Psi)$.

Let $\Sigma' \subseteq \Sigma$ *with* $\Sigma' \neq \emptyset$ *and* $A' \in \mathcal{L}_{\Sigma'}$ *such that* $Mod_{\Sigma'}(A') = \{\omega' \mid \omega \in Mod_\Sigma(A)\}$ *where* ω' *is the assignment of the variables in* Σ' *as in* ω.

(Marg) $\xi_{\Sigma',A'}(\Psi_{|\Sigma'}) = \xi_{\Sigma,A}(\Psi)_{|\Sigma'}$

Let $F \in \mathcal{L}_\Sigma$ *with* $Mod_\Sigma(F) \cap dom(\Psi) \neq \emptyset$.

(Cond) $\xi_{\Sigma,A\wedge F}(\Psi|F) = \xi_{\Sigma,A}(\Psi)|F$

The postulate (IE) requires *inferential equivalence* and states that the epistemic state mapping may not change the set of conditionals accepted by an epistemic state. The epistemic state and its mapping induce the same inference relation with respect to conditionals. This is a quite strong postulate, the postulates (wIE$^\Rightarrow$) and (wIE$^\Leftarrow$) are weaker versions of (IE). Postulate (wIE$^\Rightarrow$) states that an epistemic state mapping may not remove conditionals from the set of inferred conditionals. Postulate (wIE$^\Leftarrow$) states that after an epistemic state mapping, we may not accept additional conditionals.

The postulate (Ord) expresses the postulate (IE) in terms of the induced total preorders of the epistemic states. Analogously (wOrd$^\Rightarrow$) and (wOrd$^\Leftarrow$) represent (wIE$^\Rightarrow$) and (wIE$^\Leftarrow$), respectively.

(SynSplit) states that an epistemic state mapping should preserve syntax splittings of the epistemic state. (SynSplit$^\mathrm{b}$) is a special case of (SynSplit) for syntax splittings in two sub-signatures.

The postulate (Marg) ensures the compatibility of an epistemic state mapping with marginalization. It states that changing the order in which marginalization and the epistemic state mapping are applied does not matter. This postulate is illustrated in Figure 1a. Similarly, the postulate (Cond) ensures the compatibility of an epistemic state mapping with conditionalization. (Cond) is illustrated in Figure 1b.

It is easy to see that (IE) is equivalent to the conjunction of (wIE$^\Rightarrow$) and (wIE$^\Leftarrow$) and that (Ord) is equivalent to the conjunction of (wOrd$^\Rightarrow$) and (wOrd$^\Leftarrow$). Other relationships among the postulates, such as the following, are less obvious.

Proposition 1. *The following relationships hold between the postulates:*

1. *(IE) is equivalent to (Ord).*
2. *(wIE$^\Rightarrow$) is equivalent to (wOrd$^\Rightarrow$).*
3. *(wIE$^\Leftarrow$) is equivalent to (wOrd$^\Leftarrow$).*

Proof. Let $\xi : I_1 \rightsquigarrow I_2$ be a epistemic state mapping with $I_1, I_2 \in \{TPO, OCF\}$.
Ad (2): "$\Leftarrow$" Let $(\xi_{\Sigma,A})$ satisfy (wOrd$^\Rightarrow$). Let $\Psi \in \mathcal{M}_{I_1}(\Sigma, A)$ and $\Phi = \xi_{\Sigma,A}(\Psi)$. If $\Psi \models (D|C)$, then $\min(Mod_\Sigma(CD), <_\Psi) <_\Psi \min(Mod_\Sigma(C\overline{D}), <_\Psi)$.

$$\begin{tikzcd} \mathcal{M}_I(\Sigma, A) \arrow[r, "\cdot|\Sigma'"] \arrow[d, "\xi"'] & \mathcal{M}_I(\Sigma', A) \arrow[d, "\xi"] \\ \mathcal{M}_I(\Sigma, A) \arrow[r, "\cdot|\Sigma'"'] & \mathcal{M}_I(\Sigma', A) \end{tikzcd}$$

Illustration of (Marg).

$$\begin{tikzcd} \mathcal{M}_I(\Sigma, A) \arrow[r, "\cdot|F"] \arrow[d, "\xi"'] & \mathcal{M}_I(\Sigma, AF) \arrow[d, "\xi"] \\ \mathcal{M}_I(\Sigma, A) \arrow[r, "\cdot|F"'] & \mathcal{M}_I(\Sigma, AF) \end{tikzcd}$$

Illustration of (Cond).

Fig. 2: Illustration of the postulates (Cond) and (Marg). The diagrams are supposed to commutate.

In this case, (wOrd$^{\Rightarrow}$) implies $\min(Mod_\Sigma(CD), <_\Phi) <_\Phi \min(Mod_\Sigma(C\overline{D}), <_\Phi)$. This is equivalent to $\Phi \models (C|D)$. Therefore, $(\xi_{\Sigma,A})$ satisfies (wIE$^{\Rightarrow}$).

"$\Rightarrow$" Let $(\xi_{\Sigma,A})$ satisfy (wIE$^{\Rightarrow}$). Let $\Psi \in \mathcal{M}_{I_1}(\Sigma, A)$ and $\Phi = \xi_{\Sigma,A}(\Psi)$. Let $\omega_1, \omega_2 \in \Omega$ with $\omega_1 <_\Psi \omega_2$. Then, $\Psi \models (\omega_1|\omega_1 \vee \omega_2)$. (wIE$^{\Rightarrow}$) implies that $\Phi \models (\omega_1|\omega_1 \vee \omega_2)$. Therefore, $\omega_1 <_\Phi \omega_2$. We see that $(\xi_{\Sigma,A})$ satisfies (wOrd$^{\Rightarrow}$).

Ad (3): "$\Leftarrow$" Let $(\xi_{\Sigma,A})$ satisfy (wOrd$^{\Leftarrow}$). Let $\Psi \in \mathcal{M}_{I_1}(\Sigma, A)$ and $\Phi = \xi_{\Sigma,A}(\Psi)$. If $\Phi \models (D|C)$, then $\min(Mod_\Sigma(CD), <_\Phi) <_\Phi \min(Mod_\Sigma(C\overline{D}), <_\Phi)$. In this case, (wOrd$^{\Leftarrow}$) implies $\min(Mod_\Sigma(CD), <_\Psi) <_\Psi \min(Mod_\Sigma(C\overline{D}), <_\Psi)$. This is equivalent to $\Psi \models (D|C)$. Therefore, $(\xi_{\Sigma,A})$ satisfies (wIE$^{\Leftarrow}$).

"$\Rightarrow$" Let $(\xi_{\Sigma,A})$ satisfy (wIE$^{\Leftarrow}$). Let $\Psi \in \mathcal{M}_{I_1}(\Sigma, A)$ and $\Phi = \xi_{\Sigma,A}(\Psi)$. Let $\omega_1, \omega_2 \in \Omega$ with $\omega_1 <_\Phi \omega_2$. Then, $\Phi \models (\omega_1|\omega_1 \vee \omega_2)$. (wIE$^{\Leftarrow}$) implies that $\Psi \models (\omega_1|\omega_1 \vee \omega_2)$. Therefore, $\omega_1 <_\Psi \omega_2$. We see that $(\xi_{\Sigma,A})$ satisfies (wOrd$^{\Leftarrow}$).

Ad (1): This follows from (2) and (3) as (IE) is the conjunction of (wIE$^{\Rightarrow}$) and (wIE$^{\Leftarrow}$) and (Ord) is the conjunction of (wOrd$^{\Rightarrow}$) and (wOrd$^{\Leftarrow}$). $\square$

In the next sections, we will investigate the introduced postulates further for specific combinations of I_1 and I_2.

5 Epistemic State Mappings of Different Types

In this section we investigate epistemic state mappings from TPOs to TPOs (sub-section 5.1), OCFs to OCFs (sub-section 5.2), OCFs to TPOs (sub-section 5.3), and TPOs to OCFs (sub-section 5.4) in more detail.

5.1 Mapping Total Preorders to Total Preorders

Let us first consider epistemic state mappings from TPOs to TPOs. If we want (IE) or the equivalent (Ord) to hold, we do not have much choice.

Proposition 2. *The only epistemic state mapping from TPOs to TPOs that fulfils (Ord) is the identity.*

From Proposition 2 it follows that (IE) or (Ord) imply (SynSplit), (Cond), and (Marg) for epistemic state mappings from TPOs to TPOs as the identity fulfils these postulates.

5.2 Mapping Ranking Functions to Ranking Functions

Now consider the case where we map OCFs to OCFs. For such epistemic state mappings all postulates are compatible, in the sense that all postulates can be satisfied simultaneously by some epistemic state mapping.

Proposition 3. *The epistemic state mapping $\xi : OCF \rightsquigarrow OCF, \kappa \mapsto a \cdot \kappa$ for some $a \in \mathbb{N}^+$ fulfils (Ord), (SynSplit), (Cond), and (Marg).*

The epistemic state mappings in Proposition 3 are the only epistemic state mappings fulfilling all postulates.

Proposition 4. *All epistemic state mappings $\xi : OCF \rightsquigarrow OCF$ that fulfil (Ord) and (Cond) have the form $\kappa \mapsto a \cdot \kappa$ for some $a \in \mathbb{N}^+$.*

Proof. Let $\xi : OCF \rightsquigarrow OCF$ be an epistemic state mapping. First, we consider OCFs with two worlds in their domain. Let ω_1, ω_2 be worlds and $\kappa_1 : \{\omega_1, \omega_2\} \to \mathbb{N}_0,\ \{\omega_1 \mapsto 0, \omega_2 \mapsto 1\}$. Let $\kappa'_1 = \xi(\kappa_1)$. Let $a = \kappa'_1(\omega_2) - \kappa'_1(\omega_1)$. Because of (Ord), we have $a > 0$.

Let ω_3, ω_4 be worlds with $\{\omega_1, \omega_2\} \cap \{\omega_3, \omega_4\} = \emptyset$. We show that for $\kappa_2 : \{\omega_3, \omega_4\} \to \mathbb{N}_0,\ \{\omega_3 \mapsto 0, \omega_4 \mapsto b\}$ with $b \in \mathbb{N}^+$ and $\kappa'_2 = \xi(\kappa_2)$ it holds that $\kappa'_2(\omega_4) - \kappa'_2(\omega_3) = a \cdot b$ by induction over b.
Base Case: Let $b = 1$. Consider the OCF $\kappa_3 : \{\omega_1, \omega_2, \omega_3, \omega_4\} \to \mathbb{N}_0,\ \{\omega_1, \omega_3 \mapsto 0, \omega_2, \omega_4 \mapsto 1\}$. Let $\kappa'_3 = \xi(\kappa_3)$. (Ord) requires that $\kappa'_3(\omega_1) = \kappa'_3(\omega_3)$ and $\kappa'_3(\omega_2) = \kappa'_3(\omega_4)$. (Cond) requires that $\kappa'_3(\omega_2) - \kappa'_3(\omega_1) = a$. Therefore, $\kappa'_3(\omega_4) - \kappa'_3(\omega_3) = a = a \cdot b$. With (Cond) follows that $\kappa'_2(\omega_4) - \kappa'_2(\omega_3) = a \cdot b$.
Induction Step: Let $b = n+1$. Let $\omega_5 \notin \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be an additional world. Consider the OCF $\kappa_4 : \{\omega_3, \omega_4, \omega_5\} \to \mathbb{N}_0,\ \{\omega_3 \mapsto 0, \omega_5 \mapsto n, \omega_4 \mapsto n+1\}$. Let $\kappa'_4 = \xi(\kappa_4)$. The induction hypothesis in combination with (Cond) requires that $\kappa'_4(\omega_5) - \kappa'_4(\omega_3) = a \cdot n$ and $\kappa'_4(\omega_4) - \kappa'_4(\omega_5) = a$. Hence, $\kappa'_4(\omega_4) - \kappa'_4(\omega_3) = a \cdot (n+1)$. With (Cond) follows that $\kappa'_2(\omega_4) - \kappa'_2(\omega_3) = a \cdot b$.

The statement proven by induction is extended to situations with $\{\omega_1, \omega_2\} \cap \{\omega_3, \omega_4\} \neq \emptyset$ by comparing OCFs over $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ with OCFs over $\{\omega_5, \omega_6\}$ for additional worlds ω_5, ω_6 as in the base case of the induction.

Now, consider any restricted ranking function κ. Let $\omega \in dom(\kappa)$ and $\omega_0 \in \kappa^{-1}(0)$ and $\kappa' = \xi(\kappa_5)$. Because of (Ord) and because the minimal rank of an OCF is 0, we have that $\kappa'(\omega_0) = 0$. The result above and (Cond) yields $\kappa'(\omega) = \kappa'(\omega) - \kappa'(\omega_0) = a \cdot (\kappa(\omega) - \kappa(\omega_0)) = a \cdot \kappa(\omega)$. $\square$

5.3 Mapping Ranking Functions to Total Preorders

In this sub-section, we want to investigate epistemic state mappings from OCFs to TPOs on worlds.

Proposition 5 (τ^*). *There is a unique epistemic state mapping $\tau^* : OCF \rightsquigarrow TPO$ fulfilling (IE). This mapping is $\tau^* : OCF \rightsquigarrow TPO, \quad \kappa \mapsto \preceq_\kappa$.*

The mapping τ^* is surjective: For a given TPO $\preceq$ it is easy to construct an OCF κ such that $\preceq\, = \tau^*(\kappa)$. But τ^* is not injective as there are more ranking functions than TPOs for any given (non-empty) signature. Furthermore, τ^* satisfies the other introduced postulates for epistemic state mappings.

Proposition 6. *τ^* fulfils (SynSplit), (Marg), and (Cond).*

Hence, (Ord) implies (SynSplit), (Marg), and (Cond) for epistemic state mappings from OCFs to TPOs.

5.4 Mapping Total Preorders to Ranking Functions

Finally, we consider epistemic state mappings that map a TPO to an OCF. Since the functions in τ^* are not bijective, we cannot simply reverse them. On the contrary, there is more than one epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils (Ord). That is not surprising as an OCF contains more information than a TPO over the same domain. The additional information is the absolute distance between worlds. The functions in ρ need to fill in this missing information.

Example 2. Let $\rho : TPO \rightsquigarrow OCF$ be an epistemic state mapping defined as follows. For $\preceq\, \in \mathcal{M}_{TPO}(\Sigma, A)$ let $L_0^{\preceq} = \min(dom(\preceq), \preceq)$ and $L_k^{\preceq} = \min(dom(\preceq) \setminus (L_0^{\preceq} \cup \cdots \cup L_{k-1}^{\preceq}), \preceq)$. Every set $L_k^{\preceq}$ corresponds to the k-th layer of $\preceq$. The sets $L_i^{\preceq}$ and $L_j^{\preceq}$ are disjunct for $i \neq j$. We define $\xi(\preceq) = \kappa$ with $\kappa(\omega) = k$ such that $\omega \in L_k^{\preceq}$ for every $\omega \in dom(\preceq)$. For example, the TPO $ab \prec \overline{a}b, a\overline{b} \prec \overline{a}\overline{b}$ over $\Sigma = \{a, b\}$ is mapped to $\kappa : \{ab \mapsto 0, \overline{a}b \mapsto 1, a\overline{b} \mapsto 1, \overline{a}\overline{b} \mapsto 2\}$ by ρ.

This epistemic state mapping ρ fulfils (Ord).

To limit the possible outcomes of the transformation, we consider additional postulates such as (SynSplit). Unfortunately, there is no epistemic state mapping ρ that fulfils both (Ord) and (SynSplit).

Proposition 7. *There is no epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils (Ord) and (SynSplit).*

This incompatibility persists if we consider the weaker (SynSplit$^{\mathrm{b}}$) instead of (SynSplit) and (wIE$^{\Rightarrow}$) instead of (IE).

Proposition 8. *There is no epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils both (wIE$^{\Rightarrow}$) and (SynSplit^b).*

Proof. Let $\Sigma = \{a, b, c, d\}$ be a signature and $\preceq$ be the TPO over Σ displayed in Figure 3. This TPO has the syntax splitting $\{a, b\} \,\dot\cup\, \{c, d\}$. Assume there is an OCF κ with syntax splitting $\{a, b\} \,\dot\cup\, \{c, d\}$ such that $\omega_1 \prec \omega_2$ implies $\kappa(\omega_1) < \kappa(\omega_2)$. Then there are OCFs $\kappa_1 : \Omega_{\{a,b\}} \to \mathbb{N}_0$ and $\kappa_2 : \Omega_{\{c,d\}} \to \mathbb{N}_0$ such that $\kappa = \kappa_1 \oplus \kappa_2$. Let $\kappa_1(ab) = 0$, $\kappa_1(a\overline{b}) = i$, $\kappa_1(\overline{a}b) = j$, $\kappa_1(\overline{a}\overline{b}) = k$, $\kappa_2(cd) = 0$, $\kappa_2(c\overline{d}) = l$, $\kappa_2(\overline{c}d) = m$, and $\kappa_2(\overline{c}\overline{d}) = n$. As $\omega_1 \prec \omega_2$ implies $\kappa(\omega_1) < \kappa(\omega_2)$ for every $\omega_1, \omega_2 \in \Omega_\Sigma$ we have that

$$m + j = \kappa_1(\overline{a}b) + \kappa_2(\overline{c}d) = \kappa(\overline{a}b\overline{c}d) < \kappa(a\overline{b}\overline{c}\overline{d}) = \kappa_1(a\overline{b}) + \kappa_2(\overline{c}\overline{d}) = i + n.$$

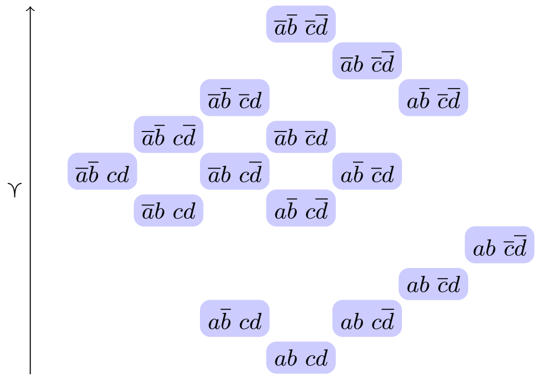


Fig. 3: Total preorder $\preceq$ on $\Sigma = \{a, b, c, d\}$ with syntax splitting $\{a, b\} \mathbin{\dot\cup} \{c, d\}$. There is no OCF with that syntax splitting that induces a superset of $\preceq$.

$$
\begin{tikzcd}
\mathcal{M}_{OCF}(\Sigma, A) \arrow[r, "\cdot_{|\Sigma'}"] & \mathcal{M}_{OCF}(\Sigma', A) \arrow[d, "\tau^*"] \\
\mathcal{M}_{TPO}(\Sigma, A) \arrow[u, "\rho"] \arrow[r, "\cdot_{|\Sigma'}"'] & \mathcal{M}_{TPO}(\Sigma', A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_{OCF}(\Sigma, A) \arrow[r, "\cdot|F"] & \mathcal{M}_{OCF}(\Sigma, AF) \arrow[d, "\tau^*"] \\
\mathcal{M}_{TPO}(\Sigma, A) \arrow[u, "\rho"] \arrow[r, "\cdot|F"'] & \mathcal{M}_{TPO}(\Sigma, AF)
\end{tikzcd}
$$

Fig. 5: Illustration of Proposition 9. Both diagrams commutate. Precondition of both propositions is that $\rho : TPO \rightsquigarrow OCF$ fulfils (Ord).

Analogously, we get $j > n$ from $\kappa(\overline{a}bcd) > \kappa(ab\overline{c}\overline{d})$ and $m > i$ from $\kappa(ab\overline{c}d) > \kappa(a\overline{b}cd)$. The combination of these inequations is a contradiction. The assumed ranking function κ cannot exist. $\square$

The combination of (wIE$^{\Leftarrow}$) and (SynSplit) is consistent, as we will see later.

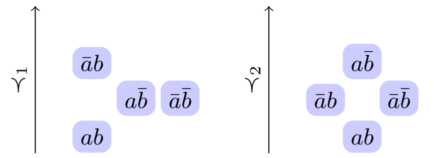
Any epistemic state mapping ρ from TPOs to OCFs satisfying (Ord) is compatible with τ^* (Prop. 5) with respect to marginalization and conditionalization.

Proposition 9. *Let $\rho : TPO \rightsquigarrow OCF$ be an epistemic state mapping that fulfils (Ord). For every TPO $\preceq \in \mathcal{M}_{TPO}(\Sigma, A)$ and $\Sigma' \subseteq \Sigma$ and $F \in \mathcal{L}_\Sigma$ it holds that $\tau^*(\rho(\preceq)_{|\Sigma'}) = \preceq_{|\Sigma'}$ and $\tau^*(\rho(\preceq)|F) = \preceq|F$.*

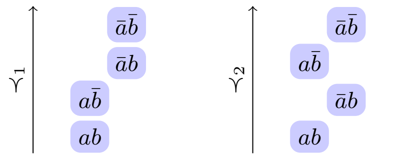
It would be useful if a transformation from a TPO to an OCF preserved marginalization and conditionalization in the way τ^* does for the other direction. But Postulate (Cond) is unfulfillable in combination with (Ord). (Cond) is even incompatible with the weaker Postulate (wIE$^{\Rightarrow}$).

Proposition 10. *There is no epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils (Cond) and (wIE$^{\Rightarrow}$).*

Proof. Let $\Sigma = \{a, b\}$ be a signature and $\preceq_1, \preceq_2$ be the TPO over Σ displayed in Figure 6a. We have $\preceq_1|a = \preceq_2|a$ and $\preceq_1|b = \preceq_2|b$. Let $\kappa_1 = \rho(\preceq_1)$ and



TPOs $\preceq_1$ and $\preceq_2$ on $\Sigma = \{a, b\}$ showing that (Cond) is incompatible with (wIE$^{\Rightarrow}$) for mappings from TPOs to OCFs in the proof of Proposition 10.



TPOs $\preceq_1$ and $\preceq_2$ on $\Sigma = \{a, b\}$ showing that (Marg) is incompatible with (wIE$^{\Rightarrow}$) for mappings from TPOs to OCFs in the proof of Proposition 11.

Fig. 7: Counterexamples in the proofs of Propositions 10 and 11.

$\kappa_2 = \rho(\preceq_2)$. If (wIE$^{\Rightarrow}$) and (Cond) were true it would imply $\kappa_1(\bar{a}b) = \kappa_1|b(\bar{a}b) = \kappa_2|b(\bar{a}b) = \kappa_2(\bar{a}b)$ and $\kappa_1(a\bar{b}) = \kappa_1|a(a\bar{b}) = \kappa_2|a(a\bar{b}) = \kappa_2(a\bar{b})$. This contradicts (wIE$^{\Rightarrow}$) as (wIE$^{\Rightarrow}$) requires $\kappa_1(\bar{a}b) > \kappa_1(a\bar{b})$ and $\kappa_2(\bar{a}b) < \kappa_2(a\bar{b})$. $\square$

(Marg) is also unfulfillable in combination with (Ord) or (wIE$^{\Rightarrow}$) in general.

Proposition 11. *There is no epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils (wIE$^{\Rightarrow}$) and (Marg).*

Proof. Let $\Sigma = \{a, b\}$ be a signature and $\preceq_1, \preceq_2$ be the total preorders over Σ displayed in Figure 6b. Let $\Sigma_1 = \{a\}$ and $\Sigma_2 = \{b\}$. We have $\preceq_{1|\Sigma_1} = \preceq_{2|\Sigma_1}$ and $\preceq_{1|\Sigma_2} = \preceq_{2|\Sigma_2}$. Let $\kappa_1 = \rho(\preceq_1)$ and $\kappa_2 = \rho(\preceq_2)$. If (wIE$^{\Rightarrow}$) and (Marg) were true it would imply $\kappa_1(\bar{a}b) = \kappa_{1|\Sigma_1}(\bar{a}b) = \kappa_{2|\Sigma_1}(\bar{a}b) = \kappa_2(\bar{a}b)$ and $\kappa_1(a\bar{b}) = \kappa_{1|\Sigma_2}(a\bar{b}) = \kappa_{2|\Sigma_2}(a\bar{b}) = \kappa_2(a\bar{b})$. This contradicts (wIE$^{\Rightarrow}$) as (wIE$^{\Rightarrow}$) requires $\kappa_1(\bar{a}b) > \kappa_1(a\bar{b})$ and $\kappa_2(\bar{a}b) < \kappa_2(a\bar{b})$. $\square$

The Propositions 8, 10, and 11 all showed that (wIE$^{\Rightarrow}$) in combination with some of the other postulates cannot be fulfilled. (wIE$^{\Leftarrow}$) on the other hand can be fulfilled in combination with these other postulates. But the following triviality result shows that there is only one epistemic state mapping fulfilling the combination of (wIE$^{\Leftarrow}$) and (Cond).

Proposition 12. *The only epistemic state mapping $\rho : TPO \rightsquigarrow OCF$ that fulfils (wIE$^{\Leftarrow}$) and (Cond) maps every TPO to the uniform ranking function κ_{uni}.*

Proof. Let ρ be an epistemic state mapping fulfilling (wIE$^{\Leftarrow}$) and (Marg). Let Σ be a signature and $\omega_1, \omega_2 \in \Omega_\Sigma$ with $\omega_1 \neq \omega_2$. Choose a third world $\omega_3 \in \Omega_\Sigma$ with $\omega_3 \notin \{\omega_1, \omega_2\}$ and consider the TPOs $\omega_3 \prec_1 \omega_2 \prec_1 \omega_1 \prec_1 \omega_4, \ldots, \omega_n$ and $\omega_3 \prec_2 \omega_1 \prec_2 \omega_2 \prec_2 \omega_4, \ldots, \omega_n$ with $\{\omega_4, \ldots, \omega_n\} = \Omega_\Sigma \setminus \{\omega_1, \omega_2, \omega_3\}$. Let $\kappa_1 = \rho(\preceq_1)$ and $\kappa_2 = \rho(\preceq_2)$. The postulate (wIE$^{\Leftarrow}$) requires that

$$\kappa_1(\omega_2) \leqslant \kappa_1(\omega_1) \quad \text{and} \quad \kappa_2(\omega_1) \leqslant \kappa_2(\omega_2). \tag{$*$}$$

Let $A = \omega_3 \vee \omega_1$ and $B = \omega_3 \vee \omega_2$. Conditionalization yields $\preceq'_A = \preceq_1|A = \preceq_2|A$ and $\preceq'_B = \preceq_1|B = \preceq_2|B$. Postulate (Cond) requires $\kappa_1|A = \rho(\preceq'_A) = \kappa_2|A$ and

$\kappa_1|B = \rho(\preceq'_B) = \kappa_2|B$. This implies $\kappa_1(\omega_1) = \kappa_2(\omega_1)$ and $\kappa_1(\omega_2) = \kappa_2(\omega_2)$. With $(*)$ it follows that $\kappa_1(\omega_2) \leqslant \kappa_1(\omega_1) = \kappa_2(\omega_1) \leqslant \kappa_2(\omega_2) = \kappa_2(\omega_2)$. Therefore, we can replace the $\leqslant$ in this chain of (in-)equations by $=$. Let $C = \omega_1 \vee \omega_2$. We can see that both $\preceq_1|C = \{\omega_1 \prec \omega_2\}$ and $\preceq_2|C = \{\omega_2 \prec \omega_1\}$ are mapped to the uniform ranking function κ_{uni} due to (Cond).

Since we can choose any two worlds as ω_1, ω_2 in this argumentation, (Cond) requires that any TPO is mapped to the uniform ranking function κ_{uni}. This mapping to κ_{uni} fulfils all three postulates. □

Note that these results only apply to epistemic state mappings defined for all TPOs. Mappings that are defined over a certain subset of TPOs might still fulfil combinations of postulates.

6 Conclusion

In this paper, we introduced the notion of epistemic state mappings, i.e., mappings within and across the frameworks of OCFs and TPOs. We proposed postulates for these mappings that ensure the preservation of certain properties of the epistemic state across the mapping. The properties considered include the set of entailed conditionals and syntax splitting. Other postulates ensure compatibility with the operations marginalization and conditionalization, respectively. Furthermore, we investigated the relationships among the proposed postulates in general and for each combination of the considered framework. Some postulates are entailed by other postulates, e.g., (SynSplit) entails (SynSplit$^{\text{b}}$), (IE) is equivalent to (Ord). We also showed that there are combinations of the postulates which cannot be satisfied simultaneously, e.g., there is no epistemic state mapping from TPOs to OCFs that fulfils both (wIE$^{\Rightarrow}$) and (SynSplit$^{\text{b}}$). In some cases we identified all possible epistemic state mapping satisfying a certain combination of postulates, e.g., the only mapping from TPOs to OCFs that fulfils both (wIE$^{\Rightarrow}$) and (SynSplit$^{\text{b}}$) is the trivial mapping of every TPO to κ_{uni}.

Our current work includes extending the investigation of epistemic state mappings and their properties for establishing further relationships between OCFs and TPOs. Furthermore, we will consider epistemic state mappings among particular subclasses of TPOs and OCFs. We expect to find interesting and relevant subclasses such that epistemic state mappings over these subclasses fulfil combinations of postulates that are not fulfilled by epistemic state mappings over the full sets of TPOs and OCFs.

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