A Galois Framework for the Study of Analogical Classifiers

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Abstract

In this paper, we survey some recent advances in the study of analogical classifiers, *i.e.,* classifiers that are compatible with the principle of analogical inference. We will present a Galois framework induced by relation between formal models of analogy and the corresponding classes of analogy preserving functions. The usefulness these general results will be illustrated over Boolean domains, which explicitly present the Galois closed sets of analogical classifiers for different pairs of formal models of Boolean analogies.

Keywords

Analogical proportion, analogical reasoning, analogical classifier, Galois theory

1. Motivation and Background

Analogical reasoning (AR) is a remarkable capability of human thought that exploits parallels between situations of different nature to infer plausible conclusions, by relying simultaneously on similarities and dissimilarities. Machine learning (ML) and artificial intelligence (AI) have tried to develop AR, mostly based on cognitive considerations, and to integrate it in a variety of ML tasks, such as natural language processing (NLP), preference learning and recommendation [1, 2, 3, 4, 5]. Also, analogical extrapolation (inference) can solve difficult reasoning tasks such as *scholastic aptitude tests* and *visual question answering* [6, 7, 8, 9]. Inference based on AR can also support dataset augmentation (analogical extension and extrapolation) for model learning, especially in environments with few labeled examples [10]. Furthermore, AR can also be performed at a meta level for transfer learning [11, 12] where the idea is to take advantage of what has been learned on a source domain in order to improve the learning process in a target domain related to the source domain. Moreover, analogy making can provide useful explanations that rely on the parallel example-counterexample [13] and guide counterfactual generation [14].

However, early works lacked theoretical and formalizational support. The situation started to change about a decade ago when researchers adopted the view of analogical proportions

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as statements of the form "a relates to b as c relates to d", usually denoted $a : b :: c : d$. Such proportions are at the root of the analogical inference mechanism, and several formalisms to study this mechanism have been proposed, which follow different axiomatic and logical approaches [15, 16]. For instance, [17] introduces the following 4 postulates in the linguistic context as a guideline for formal models of analogical proportions: *symmetry* (if $a : b :: c : d$, then $c : d :: a : b$), *central permutation* (if $a : b :: c : d$, then $a : c :: b : d$), *strong inner reflexivity* (if $a : a :: c : d$, then $d = c$), and *strong reflexivity* (if $a : b :: a : d$, then $d = b$). Such postulates appear reasonable in the word domain, but they can be criticized in other application domains. For instance, in a setting where two distinct conceptual spaces are involved, as in $wine : French :: beer : Belgian$ where two different spaces "drinks" and "nationality" are considered, the central permutation is not tolerable.

Recently, [18] proposed an algebraic framework of analogies that is naturally embedded into first-order logic via model-theoretic types. It provides a unifying setting where the different axiomatic approaches in the literature and respective domains of interpretation can be considered.

Example 1. Among the classical models of analogy on the two-element set $\{0, 1\}$, it is noteworthy to mention [19] and [20] definitions of Boolean analogy that correspond respectively to the relations R_1 and R_2 below. In this paper we represent a relation as a matrix whose columns are precisely the tuples belonging to the relation.

$$R_1 := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R_2 := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Note that R_2 contains only patterns of the form $x : x :: y : y$ and $x : y :: x : y$, and it is often referred to as the *minimal model* of analogy.

Other approaches to formalizing the notion of analogy, include the *factorial* view of [21] and the *functional* view of [22, 23], except that it is not bound by the central permutation postulate. Such a framework is close to Gentner's symbolic model of analogical reasoning [24] based on *structure mapping theory* and first implemented in [25]. Note that different axiomatic approaches entail different dataset augmentation procedures, and may impact differently on several tasks related to AR.

A key task associated with AR is *analogy solving*, i.e. finding or extrapolating, for a given triple a, b, c a value x such that $a : b :: c : x$ is a valid analogy. Such a task has been addressed in the framework of case-based reasoning (CBR), where solutions are generated by *retrieval* and *adaptation* [26, 27]. Following the same tracks, AR was also adapted to analogy based classification [28] where objects are viewed as attribute tuples (instances) $\mathbf{x} = (x_1, \ldots, x_n)$. Indeed, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in analogical proportion for most of their attributes, and class labels are known for $\mathbf{a}, \mathbf{b}, \mathbf{c}$ but unknown for $\mathbf{d}$, then one may infer the label for $\mathbf{d}$ as a solution of an analogical proportion equation. All these applications rely on the same idea: if four instances $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are in analogical proportion for most of the attributes describing them, then it may still be the case for the other attributes $f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})$ (for some function f). This principle is called *analogical inference principle* (AIP).

Theoretically, it is quite challenging to find and characterize situations where AIP can be soundly applied. A first step toward explaining the analogical mechanism consists in characterizing the set of functions f for which AIP is sound (*i.e.*, no error occurs) no matter which triplets of examples are used. In case of Boolean attributes and for the minimal model R_2, it was shown in [10] that these so-called "analogy-preserving" (AP) functions coincide exactly with the set of affine Boolean functions. Moreover, it was also shown that, when the function is not affine, the prediction accuracy remains high if the function is close to being affine [29]. These results were later extended to nominal (finite) underlying sets when taking the minimal model of analogy in both the domain and codomain of classifiers [30].

Intuitively, this class will change when adopting different models of analogy. This motivated a deeper study of the relation between formal models of analogy and the corresponding class of AP functions, and which culminated in a *Galois theory of analogical classifiers* [31]. In this paper, we briefly survey this Galois framework for analogical classifiers, and we illustrate these results for Boolean classifiers by revisiting different formal Boolean models of analogy, and describe the corresponding Galois closed sets of analogical classifiers.

This paper is organized as follows. We first briefly survey the universal algebraic framework pertaining to relational preservation in Section 2. We then adapt the latter to the framework of analogical preservation in Section 3, and recall the Galois theory for analogical classifiers presented in [31]. We illustrate these results by considering two classical models of Boolean analogies and describing sets of analogical classifiers accordingly. As a by-product, it follows that they actually correspond to the set of affine Boolean functions.

2. Galois Theories for Functions

Let A and B be nonempty sets. A *function of several arguments* from A to B is a mapping $f\colon A^n \to B$ for some natural number n called the *arity* of f. Denote by $\mathcal{F}_{AB}^{(n)}$ the set of all n-ary functions of several arguments from A to B, and let $\mathcal{F}_{AB} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_{AB}^{(n)}$.

In the case when $A = B$ we speak of *operations* on A, and we use the notation $\mathcal{O}_A^{(n)} := \mathcal{F}_{AA}^{(n)}$ and $\mathcal{O}_A := \mathcal{F}_{AA}$. For any set $C \subseteq \mathcal{F}_{AB}$, the *$n$-ary part* of C is $C^{(n)} := C \cap \mathcal{F}_{AB}^{(n)}$. If $f \in \mathcal{F}_{BC}^{(n)}$ and $g_1, \dots, g_n \in \mathcal{F}_{AB}^{(m)}$, then the *composition* $f(g_1, \dots, g_n)$ belongs to $\mathcal{F}_{AC}^{(m)}$ and is defined by the rule

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})) \quad \text{for all } \mathbf{a} \in A^m.$$

The i-th n-ary *projection* $\mathrm{pr}_i^{(n)} \in \mathcal{O}_A^{(n)}$ is defined by $\mathrm{pr}_i^{(n)}(a_1, \dots, a_n) := a_i$ for all $a_1, \dots, a_n \in A$. We denote by $\mathcal{J}_A$ the set of all projections on A.

The notion of functional composition can be extended to sets of functions as follows. Let $C \subseteq \mathcal{F}_{BC}$ and $K \subseteq \mathcal{F}_{AB}$. The *composition* of C with K is the set

$$CK := \{f(g_1, \dots, g_n) \in \mathcal{F}_{AC} \mid f \in C^{(n)},\, g_1, \dots, g_n \in K^{(m)}\}.$$

A *clone* on A is a set $C \subseteq \mathcal{O}_A$ that is closed under composition and contains all projections, in symbols, $CC \subseteq C$ and $\mathcal{J}_A \subseteq C$. For $F \subseteq \mathcal{O}_A$, we denote by $\langle F \rangle$ the clone generated by F, i.e., the smallest clone on A containing F.

We say that $f \in \mathcal{F}_{AB}^{(n)}$ is a *minor* of $g \in \mathcal{F}_{AB}^{(m)}$, $f \leq g$, if $f \in \{g\}\mathcal{J}_A$. The minor relation $\leq$ is a quasi-order (a reflexive and transitive relation) on $\mathcal{F}_{AB}$. Downsets of $(\mathcal{F}_{AB}, \leq)$ are called *minor-closed* classes or *minions.* Equivalently, a set $C \subseteq \mathcal{F}_{AB}$ is a minion if $C\mathcal{J}_A \subseteq C$. A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is m*-locally closed* if for all $f \in \mathcal{F}_{AB}$ (say f is n-ary), it holds that $f \in C$ whenever for every finite subset $S \subseteq A^n$ of size at most m, there exists a $g \in \mathcal{C}$ such that $f|_S = g|_S$. A set $\mathcal{C}$ is said to be *locally closed* if it is m-locally closed for every positive integer m.

Subsets of A^m are called m-ary *relations* on A. Denote by $\mathcal{R}_A^{(m)}$ the set of all m-ary relations on A, and let $\mathcal{R}_A := \bigcup_{m \in \mathbb{N}} \mathcal{R}_A^{(m)}$. Let $f \in \mathcal{O}_A^{(n)}$ and $R \in \mathcal{R}_A^{(m)}$. We say that the function f *preserves* the relation R (or f is a *polymorphism* of R, or R is an *invariant* of f), and we write $f \rhd R$, if for all $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in R$, where $f(\mathbf{a}_1, \dots, \mathbf{a}_n)$ denotes the componentwise application of f to the tuples $\mathbf{a}_i = (a_{i1}, \dots, a_{im})$, *i.e.:*

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) := (f(a_{11}, \dots, a_{n1}), \dots, f(a_{1m}, \dots, a_{nm})).$$

The preservation relation $\rhd$ induces a Galois connection between the sets $\mathcal{O}_A$ and $\mathcal{R}_A$ of operations and relations on A. Its polarities are the maps $\operatorname{Pol}\colon \mathcal{P}(\mathcal{R}_A) \to \mathcal{P}(\mathcal{O}_A)$ and $\operatorname{Inv}\colon \mathcal{P}(\mathcal{O}_A) \to \mathcal{P}(\mathcal{R}_A)$ given by the following rules: for all $\mathcal{R} \subseteq \mathcal{R}_A$ and $\mathcal{F} \subseteq \mathcal{O}_A$,

$$\begin{aligned} \operatorname{Pol} \mathcal{R} &:= \{f \in \mathcal{O}_A \mid \forall R \in \mathcal{R} \colon f \rhd R\}, \\ \operatorname{Inv} \mathcal{F} &:= \{R \in \mathcal{R}_A \mid \forall f \in \mathcal{F} \colon f \rhd R\}. \end{aligned}$$

Under this Galois connection, the closed sets of operations are precisely the locally closed clones. The closed sets of relations, known as *relational clones,* are precisely the locally closed sets of relations that contain the empty relation and the diagonal relations and are closed under formation of primitively positively definable relations. This was first shown for finite base sets in [32, 33, 34] and later extended for arbitrary sets in [35, 36].

The preservation relation can be adapted for functions of several arguments from A to B; we now need to consider pairs of relations. Let

$$\mathcal{R}_{AB}^{(m)} := \mathcal{R}_A^{(m)} \times \mathcal{R}_B^{(m)} \quad \text{and} \quad \mathcal{R}_{AB} := \bigcup_{m \in \mathbb{N}} \mathcal{R}_{AB}^{(m)}$$

be the set of all (m-ary) *relational constraints* from A to B.

Let $f \in \mathcal{F}_{AB}^{(n)}$ and $(R, S) \in \mathcal{R}_{AB}^{(m)}$. We say that f *preserves* (R, S) (or f is a *polymorphism* of (R, S), or (R, S) is an *invariant* of f), and we write $f \rhd (R, S)$, if for all $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in S$. As before, the preservation relation $\rhd$ induces a Galois connection between the sets $\mathcal{F}_{AB}$ and $\mathcal{R}_{AB}$ of functions and relational constraints from A to B. Its polarities are the maps $\operatorname{Pol}\colon \mathcal{P}(\mathcal{R}_{AB}) \to \mathcal{P}(\mathcal{F}_{AB})$ and $\operatorname{Inv}\colon \mathcal{P}(\mathcal{F}_{AB}) \to \mathcal{P}(\mathcal{R}_{AB})$ given by the following rules: for all $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ and $\mathcal{F} \subseteq \mathcal{F}_{AB}$,

$$\begin{aligned} \operatorname{Pol} \mathcal{Q} &:= \{f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{Q} \colon f \rhd (R, S)\}, \\ \operatorname{Inv} \mathcal{F} &:= \{(R, S) \in \mathcal{R}_{AB} \mid \forall f \in \mathcal{F} \colon f \rhd (R, S)\}. \end{aligned}$$

The sets $\operatorname{Pol} \mathcal{Q}$ and $\operatorname{Inv} \mathcal{F}$ are said to be *defined* by $\mathcal{Q}$ and $\mathcal{F}$, respectively. Sets of functions of the form $\operatorname{Pol} \mathcal{Q}$ for some $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ and sets of relational constraints of the form $\operatorname{Inv} \mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{F}_{AB}$ are said to be *definable* by relational constraints and functions, respectively.

The closed sets of functions under this Galois connection were described for finite base sets in [37] and later for arbitrary sets [38]. This result was refined in [39] for sets of functions definable by relations of restricted arity.

Theorem 2 ([38, 39])**.** *Let A and B be arbitrary nonempty sets, and let $C \subseteq \mathcal{F}_{AB}$.*

1. *C is definable by constraints if and only if C is a locally closed minion.*
2. *C is definable by constraints of arity m if and only if C is an m-locally closed minion.*

The closed sets of relational constraints were described in terms of closure conditions that parallel those for relational clones. The description of the dual objects of constraints on possibly infinite sets A and B was also provided in [38] and inspired by those given in [34, 35, 36, 37] and given in terms of positive primitive first-order relational definitions applied simultaneously on antecedents and consequents. Sets of constraints that are closed under such formation schemes are said to be *closed under conjunctive minors.* Moreover, every function satisfies the empty $(\varnothing, \varnothing)$ and the equality $(=_A, =_B)$ constraints, and if a function f satisfies a constraint (R, S), then f also satisfies its *relaxations* (R', S') such that $R' \subseteq R$ and $S' \supseteq S$.

As for functions, in the infinite case, we also need to consider a "local closure" condition to describe the dual closed sets of relational constraints on A and B. A set $\mathcal{Q}$ of constraints on A and B is n*-locally closed* if it contains every relaxation of its members whose antecedent has size at most n, and it is *locally closed* if it is n-locally closed for every positive integer n.

Theorem 3 ([38, 39])**.** *For arbitrary nonempty sets A and B, and let $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ be a set of relational constraints on A and B.*

1. *$\mathcal{Q}$ is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if it is locally closed, contains the binary equality and the empty constraints, and it is closed under relaxations and conjunctive minors.*
2. *$\mathcal{Q}$ is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}^{(n)}$ of n-ary functions if and only it is n-locally closed, contains the binary equality and the empty constraints, and it is closed under relaxations and conjunctive minors.*

Let $K \subseteq \mathcal{F}_{AB}$ and let C_1 and C_2 be clones on A and B. We say that K is *stable under right composition with* C_1 if $KC_1 \subseteq K$, and we say that K is *stable under left composition with* C_2 if $C_2K \subseteq K$. We say that K is (C_1, C_2)*-stable* or a (C_1, C_2)*-clonoid,* if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Motivated by earlier results on linear definability of equational classes of Boolean functions [40] which were described in terms of stability under compositions with the clone of constant preserving affine functions, [41] introduced a Galois framework for describing sets of functions $\mathcal{F} \subseteq \mathcal{F}_{AB}$ stable under right and left compositions with clones C_1 on A and C_2 on B, respectively. For that they restricted the defining dual objects to relational constraints (R, S) where R and S invariant under C_1 and C_2, respectively, i.e., $R \in \operatorname{Inv} C_1$ and $S \in \operatorname{Inv} C_2$. These were referred to as (C_1, C_2)*-constraints.* We denote by $\mathcal{R}_{AB}^{(C_1,C_2)}$ the set of all (C_1, C_2)-constraints.

Theorem 4 ([41])**.** *Let A and B be arbitrary nonempty sets, and let C_1 and C_2 clones on A and B, respectively. A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by some set of (C_1, C_2)-constraints if and only if $\mathcal{C}$ is locally closed and stable under right and left composition with C_1 and C_2, respectively, i.e., it is a locally closed (C_1, C_2)-clonoid.*

Dually, a set $\mathcal{Q}$ of (C_1, C_2)-constraints is definable by a set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if $\mathcal{Q} = \mathrm{Inv}\, \mathcal{C} \cap \mathcal{R}_{AB}^{(C_1,C_2)}$. To describe the dual closed sets of (C_1, C_2)-constraints, [41] observed that conjunctive minors of (C_1, C_2)-constraints are themselves (C_1, C_2)-constraints. However, this is not the case for relaxations. They thus proposed the following variants of local closure and of constraint relaxations.

A set $\mathcal{Q}_0$ of (C_1, C_2)-constraints is said to be (C_1, C_2)*-locally closed* if the set $\mathcal{Q}$ of all relaxations of the various constraints in $\mathcal{Q}_0$ is locally closed. A relaxation (R_0, S_0) of a relational constraint (R, S) is said to be a (C_1, C_2)*-relaxation* if (R_0, S_0) is a (C_1, C_2)-constraint.

Theorem 5 ([41]). *Let A and B be arbitrary nonempty sets, and let C_1 and C_2 clones on A and B, respectively. A set $\mathcal{Q}$ of (C_1, C_2)-constraints is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if it is (C_1, C_2)-locally closed and contains the binary equality constraint, the empty constraint, and it is closed under (C_1, C_2)-relaxations and conjunctive minors.*

In this paper, we will focus on relational constraints whose antecedent and consequent are derived from analogies, and that we will refer to as *analogical constraints*. We will denote the set of all analogical constraints from A to B by $\mathcal{A}_{AB}$.

3. Galois Theory for Analogical Classifiers

As mentioned in the Introduction, analogical inference yields competitive results in classification and recommendation tasks. However, the justification of why and when a classifier is compatible with the analogical inference principle (AIP) remained rather obscure until the work [10]. In this paper the authors considered the minimal Boolean analogy model (see R_2 in Example 1) and addressed the problem of determining those *Boolean classifiers for which the AIP always holds*, that is, for which there are no classification errors. Surprisingly, they showed that they correspond to "analogy preserving" and that they constitute the clone of affine functions. This result was later generalized to binary classification tasks on nominal (finite) domains in [30] where the authors considered the more stringent notion of "hard analogy preservation". By taking the same minimal analogy model on both the domain and the label set, the authors showed that in this case the sets of hard analogy preserving functions constitute Burle's clones [42].

Definition 6. Let A and B be sets, and let R and S be analogical proportions defined on the two sets, respectively. A function $f\colon A^n \to B$ is *analogy-preserving* (AP for short) relative to (R, S) if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in A^n$:

$$\big(R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \quad \text{and} \quad S\text{-solv}(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))\big) \quad \Longrightarrow \quad S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})),$$

where $R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is a shorthand for $(a_i, b_i, c_i, d_i) \in R$ for all $i \in \{1, \ldots, n\}$ and S-solv$(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))$ means that there is an $x \in B$ with $S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), x)$. Denote by $\mathsf{AP}(R, S)$ the set of all analogy-preserving functions relative to (R, S).

This relation between functions and formal models of analogy gives rise to a Galois connection whose closed sets of functions correspond exactly to the classes of analogical classifiers. As the following result shows, we can use the universal algebraic tools of Section 2 to investigate analogy preservation.

Proposition 7. *Let R and S be analogical proportions defined on sets A and B, respectively. Then* $\mathsf{AP}(R,S) = \mathrm{Pol}(R,S')$*, where*

$$S' := S \cup \{(a,b,c,d) \in B^4 \mid \nexists x \in B\colon (a,b,c,x) \in S\}. \tag{1}$$

Consequently, $\mathsf{AP}(R,S)$ *is a locally closed minion.*

Example 8. The derived relations as in Proposition 7 corresponding to the formal models of Boolean analogies in Example 1 are the following:

$$R_1' = R_1 \quad \text{and} \quad R_2' = R_2 \cup \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

To fully describe the sets of the form $\mathrm{Pol}(R,S')$ we need to introduce some variants of the closure conditions discussed in Section 2. Let $\mathcal{R}$ be set of m-ary relations on A. An $m \times n$ matrix D whose columns belong to a relation $R \in \mathcal{R}$, is called an $\mathcal{R}$*-locality*. Let $\mathcal{Q} \subseteq \mathcal{R}_{AB}$, and let $\mathcal{Q}_1 := \{R \in \mathcal{R}_A \mid \exists S \in \mathcal{R}_B$ such that $(R,S) \in \mathcal{Q}\}$. A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is Q*-locally closed* if for all $f \in \mathcal{F}_{AB}$ (say f is n-ary), it holds that $f \in C$ whenever for every $\mathcal{Q}_1$-locality D, either

1. there exists a $g \in \mathcal{C}$ such that $fD = gD$, or
2. for any relation R in $\mathcal{Q}_1$ such that $D \preceq R$ and for any

   $$T \in \{S \in \mathcal{R}_B \mid (R,S) \in \mathcal{Q},\, \mathcal{C}R \subseteq S\}$$

   we have that $fR \subseteq T$.

Let $\mathcal{A}'_B := \{S' \mid S \in \mathcal{A}_B\}$, and let $\mathcal{A}_{AB} := \mathcal{A}_A \times \mathcal{A}'_B$. We refer to the elements of $\mathcal{A}_{AB}$ as *analogical constraints* from A to B. The set of analogical constraints that are (C_1,C_2)-constraints will be denoted by

$$\mathcal{A}_{AB}^{(C_1,C_2)} := \mathcal{A}_{AB} \cap \mathcal{R}_{AB}^{(C_1,C_2)}.$$

A set $\mathcal{C}$ is said to be (C_1,C_2)*-analogically locally closed* if it is $\mathcal{A}_{AB}^{(C_1,C_2)}$-locally closed. Note that $\mathcal{A}_{AB} = \mathcal{A}_{AB}^{(\mathcal{J}_A,\mathcal{J}_B)}$, and in this case we simply say that $\mathcal{C}$ is *analogically locally closed.*

Theorem 9. *Let A and B be arbitrary nonempty sets, and let C_1 and C_2 be clones on A and B, respectively.*

1. *A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by analogical (C_1,C_2)-constraints if and only if it is a (C_1,C_2)-analogically locally closed (C_1,C_2)-clonoid.*
2. *A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by analogical constraints if and only if it is an analogically locally closed minion.*

4. Application: Description of Boolean Analogical Classifiers w.r.t. Example 1

In this section we illustrate the use of the Galois theory described in Section 3 to determine the classes of analogical classifiers $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j)$ for $i, j \in \{1, 2\}$ (see Example 8 and Equation (1)).

Recall that, up to permutation of arguments, the binary Boolean functions are the following: the constant 0 and 1 functions, denoted respectively by 0 and 1, the first projection $\mathrm{pr}_1 \colon (x_1, x_2) \mapsto x_1$ and its negation $\neg_1 = \overline{\mathrm{pr}_1}$, the conjunction $\wedge$ and its negation $\uparrow$, the disjunction $\vee$ and its negation $\downarrow$, the implication $\rightarrow$ and its negation $\nrightarrow$, and the addition $+$ modulo 2 and its negation $\leftrightarrow$. Note that $\uparrow$ and $\downarrow$ are often referred to as *Sheffer functions* as each one of them can generate the class of all Boolean functions by taking compositions and variable substitutions.

Observe that the constant tuples **0** and **1** belong to every R_i ($i \in \{1, 2\}$), and thus every such R_i is invariant under I, i.e., $\mathsf{I}R_i \subseteq R_i$. Hence, for every $i, j \in \{1, 2\}$, $\mathrm{Pol}(R_i, R'_j)$ is stable under right composition with I. This leads us to considering the following notion.

A function f is said to be a C-*minor* of a function g if $f \in gC$. Recall that in the particular case when $C = \mathcal{J}_{\{0,1\}}$, f is called a minor of g. The functions f and g are said to be *equivalent*, denoted by $f \equiv g$, if f is a minor of g and g is a minor of f. For further background on these notions and variants see, e.g., [43, 44, 37].

Since $\mathrm{Pol}(R_i, R'_j)$ is stable under right composition with I, this means that if an I-minor f of a function g does not belong to $\mathrm{Pol}(R_i, R'_j)$, then neither does g. This observation constitutes a main tool in describing the sets of the form $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j)$.

Proposition 10. *Let* L *denote the clone of affine functions. We have* $\mathsf{AP}(R_2, R_2) = \mathsf{AP}(R_2, R_1) = \mathsf{AP}(R_1, R_2) = \mathsf{AP}(R_1, R_1) = \mathsf{L}$.

Proof. We make use of the fact that $\mathsf{AP}(R, S) = \mathrm{Pol}(R, S')$. Since $R_1 = R'_1$, it follows immediately that $\mathrm{Pol}(R_1, R'_1) = \mathrm{Pol}\, R_1$ is a clone, and it is well known that $\mathrm{Pol}\, R_1 = \mathsf{L}$.

To prove $\mathsf{AP}(R_2, R_2) = \mathrm{Pol}(R_2, R'_2) = \mathsf{L}$, we make use of main tool given above. Since (R_2, R'_2) is a relaxation of (R_1, R'_1), $\mathrm{Pol}(R_2, R'_2) \supseteq \mathrm{Pol}(R_1, R'_1) = \mathsf{L}$. Furthermore,

- $\wedge, \vee \notin \mathrm{Pol}(R_2, R'_2)$ because

$$\wedge \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \notin R'_2 \qquad \text{and} \qquad \vee \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \notin R'_2.$$

- $\uparrow, \downarrow \notin \mathrm{Pol}(R_2, R'_2)$ because

$$\uparrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \notin R'_2, \qquad \text{and} \qquad \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \notin R'_2.$$

- $\nrightarrow, \rightarrow \notin \mathrm{Pol}(R_2, R_2')$ because

$$\nrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin R_2', \qquad \text{and} \qquad \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \notin R_2'.$$

The result now follows by observing that any function outside of L has an I-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow\}$. (For further details, see [31].) By similar arguments, it also follows that $\mathsf{AP}(R_2, R_1) = \mathsf{AP}(R_1, R_2) = \mathsf{AP}(R_1, R_1) = \mathsf{L}$. □

5. Conclusion and Perspectives

In this paper we survey a general Galois framework for studying analogical classifiers that does not depend on the underlying domains nor the formal models of analogy considered. We also illustrate its usefulness by explicitly describing sets of analogical classifiers with respect to two classical models of analogy. As future work, we intend to further explore different formal models of analogy that may be obtained by considering different algebraic signatures.

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References

[1] S. Alsaidi, A. Decker, P. Lay, E. Marquer, P.-A. Murena, M. Couceiro, A neural approach for detecting morphological analogies, in: IEEE 8th DSAA, 2021, pp. 1–10.

[2] M. A. Fahandar, E. Hüllermeier, Learning to rank based on analogical reasoning, in: AAAI-18, 2018, pp. 2951–2958.

[3] M. A. Fahandar, E. Hüllermeier, Analogical embedding for analogy-based learning to rank, in: IDA 2021, volume 12695 of *LNCS*, Springer, 2021, pp. 76–88.

[4] N. Hug, H. Prade, G. R., M. Serrurier, Analogical proportion-based methods for recommendation – first investigations, Fuzzy Sets Syst. 366 (2019) 110–132.

[5] M. Mitchell, Abstraction and analogy-making in artificial intelligence, 2021. URL: https://arxiv.org/abs/2102.10717.

[6] P. D. Turney, The latent relation mapping engine: Algorithm and experiments, J. Artif. Intell. Res. 33 (2008) 615–655.
[7] P. D. Turney, P. Pantel, From frequency to meaning: Vector space models of semantics, J. Artif. Intell. Res. 37 (2010) 141–188.
[8] F. Sadeghi, C. L. Zitnick, A. Farhadi, Visalogy: Answering visual analogy questions, in: NIPS 2015, 2015, pp. 1882–1890.
[9] J. Peyre, I. Laptev, C. Schmid, J. Sivic, Detecting unseen visual relations using analogies, in: IEEE ICCV 2019, 2019, pp. 1981–1990.
[10] M. Couceiro, N. Hug, H. Prade, G. Richard, Analogy-preserving functions: A way to extend Boolean samples, in: IJCAI, ijcai.org, 2017, pp. 1575–1581.
[11] A. Cornuéjols, P.-A. Murena, R. Olivier, Transfer learning by learning projections from target to source, in: IDA 2020, volume 12080 of *LNCS*, Springer, 2020, pp. 119–131.
[12] S. Alsaidi, A. Decker, P. Lay, E. Marquer, P.-A. Murena, M. Couceiro, On the transferability of neural models of morphological analogies, in: AIMLAI@ECML/PKDD, 2021, pp. 76–89.
[13] E. Hüllermeier, Towards analogy-based explanations in machine learning, in: MDAI 2020, volume 12256 of *LNCS*, Springer, 2020, pp. 205–217.
[14] M. T. Keane, B. Smyth, Good counterfactuals and where to find them: A case-based technique for generating counterfactuals for explainable AI (XAI), in: ICCBR 2020, volume 12311 of *LNCS*, Springer, 2020, pp. 163–178.
[15] Y. Lepage, Analogy and formal languages, Electron. Notes Theor. Comput. Sci. 53 (2001) 180–191.
[16] L. Miclet, S. Bayoudh, A. Delhay, Analogical dissimilarity: Definition, algorithms and two experiments in machine learning, J. Artif. Intell. Res. 32 (2008) 793–824.
[17] Y. Lepage, De l'analogie rendant compte de la commutation en linguistique, 2003. URL: https://tel.archives-ouvertes.fr/tel-00004372.
[18] C. Antić, Analogical proportions, 2020. URL: https://arxiv.org/abs/2006.02854.
[19] S. Klein, Culture, mysticism & social structure and the calculation of behavior, in: Proc. 5th Europ. Conf. in Artificial Intelligence (ECAI'82), 1982, pp. 141–146.
[20] L. Miclet, H. Prade, Handling analogical proportions in classical logic and fuzzy logics settings, in: ECSQARU, Springer, 2009, pp. 638–650.
[21] N. Stroppa, F. Yvon, An analogical learner for morphological analysis, in: CoNLL 2005, ACL, 2005, pp. 120–127.
[22] N. Barbot, L. Miclet, H. Prade, Analogy between concepts, Artif. Intell. 275 (2019) 487–539.
[23] P.-A. Murena, M. Al-Ghossein, J.-L. Dessalles, A. Cornuéjols, Solving analogies on words based on minimal complexity transformation., in: IJCAI, 2020, pp. 1848–1854.
[24] D. Gentner, C. Hoyos, Analogy and abstraction, Top. Cogn. Sci. 9 (2017) 672–693.
[25] B. Falkenhainer, K. D. Forbus, D. Gentner, The structure-mapping engine: Algorithm and examples, Artif. Intell. 41 (1989) 1–63.
[26] M. M. Richter, R. O. Weber, Case-based reasoning, Springer, 2016.
[27] J. Lieber, E. Nauer, H. Prade, When revision-based case adaptation meets analogical extrapolation, in: ICCBR 2021, volume 12877 of *LNCS*, Springer, 2021, pp. 156–170.
[28] M. Bounhas, H. Prade, G. Richard, Analogy-based classifiers for nominal or numerical data, IJAR 91 (2017) 36–55.
[29] M. Couceiro, N. Hug, H. Prade, G. Richard, Behavior of analogical inference w.r.t. Boolean

functions, in: IJCAI, ijcai.org, 2018, pp. 2057–2063.
[30] M. Couceiro, E. Lehtonen, L. Miclet, H. Prade, G. Richard, When nominal analogical proportions do not fail, in: SUM 2020,, volume 12322 of *LNCS*, Springer, 2020, pp. 68–83.
[31] M. Couceiro, E. Lehtonen, Galois theory for analogical classifiers, CoRR abs/2205.04593 (2022). URL: https://doi.org/10.48550/arXiv.2205.04593.
[32] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, B. A. Romov, Galois theory for Post algebras I, Kibernetika 5 (1969) 1–10.
[33] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, B. A. Romov, Galois theory for Post algebras II, Kibernetika 5 (1969) 1–9.
[34] D. Geiger, Closed systems of functions and predicates, Pacific J. Math. 27 (1968) 95–100.
[35] L. Szabó, Concrete representation of related structures of universal algebras I, Acta Sci. Math. (Szeged) 40 (1978) 175–184.
[36] R. Pöschel, Concrete representation of algebraic structures and a general Galois theory, in: H. Kautschitsch, W. B. Müller, W. Nöbauer (Eds.), Contributions to General Algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978), Johannes Heyn, Klagenfurt, 1979, pp. 249–272.
[37] N. Pippenger, Galois theory for minors of finite functions, Discrete Math. 254 (2002) 405–419.
[38] M. Couceiro, S. Foldes, On closed sets of relational constraints and classes of functions closed under variable substitutions, Algebra Universalis 54 (2005) 149–165.
[39] M. Couceiro, On Galois connections between external operations and relational constraints: arity restrictions and operator decompositions, Acta Sci. Math. 72 (2006) 15–35.
[40] M. Couceiro, S. Foldes, Definability of Boolean function classes by linear equations over $\mathbf{GF}(2)$, Discrete Appl. Math. 142 (2004) 29–34.
[41] M. Couceiro, S. Foldes, Function classes and relational constraints stable under compositions with clones, Discuss. Math., Gen. Algebra Appl. 29 (2009) 109–121.
[42] G. A. Burle, Classes of k-valued logic which contain all functions of a single variable, Diskret. Analiz, Novosibirsk 10 (1967) 3–7.
[43] E. Lehtonen, Descending chains and antichains of the unary, linear, and monotone subfunction relations, Order 23 (2006) 129–142.
[44] E. Lehtonen, Á. Szendrei, Partial orders induced by quasilinear clones, in: Contributions to General Algebra 20, Verlag Johannes Heyn, Klagenfurt, 2012, pp. 51–84.