A conjecture on independent sets and graph covers

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Abstract

In this article, I present a simple conjecture on the number of independent sets on graph covers. The conjecture implies that the partition function of a binary pairwise attractive model is greater than that of the Bethe approximation.

Key words: graph cover, independent set, Bethe approximation,

1. Terminologies

Throughout this article, $G = (V, E)$ is a finite graph with vertices V and undirected edges E. For each undirected edge of G, we make a pair of oppositely directed edges, which form a set of directed edges $\vec{E}$. Thus, $|\vec{E}| = 2|E|$.

An *M-cover* of a graph G is its M-fold covering space[1]. All M-covers are explicitly constructed using permutation voltage assignment as follows [1]. A *permutation voltage assignment* of G is a map

$$\alpha : \vec{E} \to \mathfrak{S}_M \qquad \text{s.t.} \quad \alpha(u \to v) = \alpha(v \to u)^{-1} \quad \forall uv \in E, \tag{1}$$

where $\mathfrak{S}_M$ is the permutation group of $\{1, \ldots, M\}$. Then an M-cover $\tilde{G} = (\tilde{V}, \tilde{E})$ of G is given by $\tilde{V} := V \times \{1, \ldots, M\}$ and

$$(v, k)(u, l) \in \tilde{E} \Leftrightarrow uv \in E \text{ and } l = \alpha(v \to u)(k). \tag{2}$$

If an M-cover is M copies of G then it is called *trivial M-cover* and denoted by $G^{\oplus M}$. This is obtained by identity permutations. The *natural projection*, π, from a cover $\tilde{G}$ to G is obtained by forgetting the "layer number". That is, $\pi : \tilde{V} \to V$ is given by $\pi(u, i) = u$ and $\pi : \tilde{E} \to E$ is given by $\pi((v, k)(u, l)) = vu$.

An independent set I of a graph G is a subset of V such that none of the elements in I are adjacent in G. Formally, I is an independent set iff $u, v \in I \Rightarrow uv \notin E$. The multivariate *independent set polynomial* of G is defined by

$$p(G) := \sum_{I:\ \text{independent set}} \prod_{v \in I} x_v, \tag{3}$$

with indeterminates $x_v \quad (v \in V)$.

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[1] Interpret graphs as topological spaces.

2. The conjecture

We extend the definition of the projection map π over the multivariate polynomial ring. First, let us define a map Π for each indeterminate by $\Pi(x_v) = x_{\Pi(v)}$, where $v \in \tilde{V}$. Then this is uniquely extended to the polynomial ring as a ring homomorphism; for example $\Pi(x_v + x_{v'}) = \Pi(x_v) + \Pi(x_{v'})$ and $\Pi(x_v x_{v'}) = \Pi(x_v)\Pi(x_{v'})$.

Conjecture 1. [2] *For any bipartite graph G and its M-cover $\tilde{G}$, we conjecture the following relation:*

$$\Pi(p(\tilde{G})) \preceq p(G)^M, \tag{4}$$

where Π is defined as above and the symbol $\preceq$ means the inequalities for all coefficients of monomials.

We can interpret the conjecture more explicitly as follows. For a subset U of V, define $\mathcal{I}(\tilde{G}, U) := \{I \subset \tilde{V} | I \text{ is independent set}, \pi(I) = U\}$, where $\pi(I)$ is the image of I. Since $p(G)^M = \Pi(p(G^{\oplus M}))$, the conjecture is equivalent to the following statement:

$$|\mathcal{I}(\tilde{G}, U)| \leq |\mathcal{I}(G^{\oplus M}, U)| \text{ for all } U \subset V. \tag{5}$$

Example 1. Let G be a cycle graph of length four and let $\tilde{G}$ be its 3-cover that is isomorphic to the cycle of length twelve. Then,

$$p(G) = 1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4, \tag{6}$$

$$p(\tilde{G}) = 1 + \sum_{v=1}^{4} \sum_{m=1}^{3} x_{(v,m)} + \ldots, \tag{7}$$

$$\Pi(p(\tilde{G})) = 1 + 3(x_1 + x_2 + x_3 + x_4) + \ldots. \tag{8}$$

It takes time and effort to check the conjecture, however, it is true in this case.

Remark. The above conjecture is claimed for the pair `(bipartite graph, independent set)`. I also conjecture analogous properties for `(bipartite graph, matching)`,`(graph with even number of vertices, perfect matching)` and `(graph, Eulerian set`[3]`)`.

3. Implication of the conjecture

The conjecture originates from the theory of the Bethe approximation. The *partition function* of a binary pairwise model on a graph G is

$$Z(G; \boldsymbol{J}, \boldsymbol{h}) := \sum_{\boldsymbol{s} \in \{0,1\}^V} \exp(\sum_{uv \in E} J_{uv} s_u s_v + \sum_{v \in V} h_v s_v), \tag{9}$$

[2] I have checked the conjecture for many examples by computer.
[3] A subset of edges is Eulerian if it induces a subgraph that only has vertices of degree two and zero.

where the weights $(\boldsymbol{J}, \boldsymbol{h})$ are called *interactions*.[4] It is called *attractive* if $J_{vu} \geq 0$ for all $vu \in E$. The *Bethe partition function*[5] Z_B is defined by [2]

$$Z_B := \exp\Big(-\min_q F_B(q)\Big) \tag{10}$$

$$= \limsup_{M \to \infty} < Z(\tilde{G}) >^{1/M}, \tag{11}$$

where F_B is the Bethe free energy and $< \cdot >$ is the mean with respect to the $M!^{|E|}$ covers. (Details are omitted. See [2].)

Theorem 1. *If Conjecture 1 holds, then*

$$Z \geq Z_B \tag{12}$$

holds for any binary pairwise attractive models.[6]

Proof. From (11), the assertion of the theorem is proved if we show that

$$Z(G)^M \geq Z(\tilde{G}) \tag{13}$$

for any M-cover $\tilde{G}$ of G. In the following, we see that the partition function can be written by the independent set polynomial and thus the above inequality holds under the assumption of Conjecture 1.

$$Z(G) = \sum_{\boldsymbol{s} \in \{0,1\}^V} \exp\big(\sum_{uv \in E} J_{uv} s_u s_v + \sum_{v \in V} h_v s_v\big) \tag{14}$$

$$= \sum_{\boldsymbol{s} \in \{0,1\}^V} \prod_{uv \in E} (1 + A_{uv} s_u s_v) \prod_{v \in V} \exp(h_v s_v) \tag{15}$$

$$= \sum_{S \subset E} \Big(\prod_{uv' \in S} A_{uv'} \Big) \prod_{v \in V} \big(\sum_{s_v = 0,1} s_v^{d_v(S)} \exp(h_v s_v)\big) \tag{16}$$

$$= \prod_{v \in V} \exp(h_v) \sum_{\substack{S \subset E, U \subset V \\ S \text{and} U \text{are not ``adjacent''}}} \prod_{uv \in S} A_{uv} \prod_{v \in U} B_v \tag{17}$$

$$= \prod_{v \in V} \exp(h_v) \quad p(G'; \boldsymbol{A}, \boldsymbol{B}), \tag{18}$$

where $d_v(S)$ is the number of edges in S connecting to v, $A_{uv} = \mathrm{e}^{J_{uv}} - 1, B_v = \mathrm{e}^{-h_v}$ and G' is a bipartite graph obtained by adding a new vertex on each edge of G. □

[4] In the following, for a cover $\tilde{G}$ of G, we think that interactions are naturally induced from G.

[5] This is computed from the absolute minimum of the Bethe free energy; other Bethe approximations of the partition function corresponding to local minima are smaller than Z_B.

[6] In a quite limited situation, the inequality is proved in [3].

Remark. The Bethe approximation can also be applied to the computation of the permanent of non-negative matrices. From the combinatorial viewpoint, this problem is related to the (weighted) perfect matching problem on complete bipartite graphs. Vontobel analyzed this problem and pose a conjecture analogous to Eq.(5) [4]. This conjecture implies the inequality between the permanent and its Bethe approximation, $Z \geq Z_B$, given his formula Eq.(11). The statement $Z \geq Z_B$ is, however, directly proved by Gurvits, generalizing Schrijverfs permanental inequality [5].

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